Chapter VII
Tensor Space

ABSTRACT

In this chapter, we first give the background materials for developing tensor discrimination technologies in Section 7.1. Section 7.2 introduces some basic notations in tensor space. Section 7.3 discusses several tensor decomposition methods. Section 7.4 introduces the tensor rank.

7.1 BACKGROUND

Matrix decompositions, such as the singular value decomposition (SVD), are ubiquitous in numerical analysis. One usual way to think of the SVD is that it decomposes a matrix into a sum of rank-1 matrices. In other words, an $I_1 \times I_2$ matrix A can be expressed as a minimal sum of rank-1 matrices:

$$A = u_1 \circ v_1 + u_2 \circ v_2 + \cdots + u_r \circ v_r, \tag{7.1}$$

where $u_i \in \Re^{I_1}$ and $v_i \in \Re^{I_2}$ for all $i = 1,\cdots,r$. The operator $\circ$ denotes the outer product. Thus the ijth entry of the rank-1 matrix $a \circ b$ is the product of the ith entry of a and the jth entry of b, denoted by $(a \circ b)_{ij} = a_i b_j$. Such decompositions provide possibilities to develop fundamental concepts such as the matrix rank and the approximation theory and gain a range of applications including WWW searching and mining, image processing, signal processing, medical imaging, and principal component analysis. The decompositions are well-understood mathematically, numerically, and computationally.

A tensor is a higher order generalization of a vector or a matrix. In fact, a vector is a first-order tensor and a matrix is a tensor of order two. Furthermore speaking, tensors are multilinear mapping over a set of vector spaces. If we have data in three or more dimensions, then we mean to deal with a higher-order tensor. In tensor analysis, higher-order tensor (also known as multidimensional, multiway, or n-way array) decompositions (Martin, 2004; Comon, 2002) are used in many fields and also have received considerable theoretical interest.

Different from some classical matrix decompositions, extending matrix decompositions such as the SVD to higher-order tensors has proven to be quite difficult. Familiar matrix concepts such as rank become ambiguous and more complicated. One goal of the tensor decomposition is the same as for a matrix decomposition: to rewrite the tensor as a sum of rank-1 tensors. Consider, for example, an $I_1 \times I_2 \times I_3$ tensor A. We would like to express A as the sum of rank-1 third-order tensors, that is,

$$A = u_1 \circ v_1 \circ w_1 + u_2 \circ v_2 \circ w_2 + \cdots + u_r \circ v_r \circ w_r, \tag{7.2}$$

where, $u_i \in \Re^{I_1}$, $v_i \in \Re^{I_2}$, and $w_i \in \Re^{I_3}$ for all $i = 1, \cdots, r$. Note that if a, b, c are vectors, then $(a \circ b \circ c)_{ijk} = a_i b_j c_k$.

7.2 BASIC NOTATIONS

In this section, we introduce some elementary notations and definitions needed in the later chapter.

If A is an $I_1 \times \cdots \times I_N$ tensor, then the order of A is N. The nth mode, way, or dimension of A is of size I_n $(n = 1, \cdots, N)$.

7.2.1 Tensors as Vectors

First, let us define the operators. Let $B \in \Re^{I_1 \times I_2}$. Then vec($B$) is defined as

$$vec(B) = \begin{bmatrix} B(:,1) \\ \vdots \\ B(:,I_2) \end{bmatrix} \in \Re^{I_1 I_2}, \tag{7.3}$$

where $B(:,i)$ denotes the ith column of matrix B.

In other words, the vec operator is to transform a matrix into a vector by stacking columns of matrix B.

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