Chapter 11
Combinatorics and Probability

ABSTRACT

The chapter explores the intersection of combinatorics and probability through various engaging problems and real-world applications. It delves into classic mathematical puzzles such as the Monty Hall problem, the birthday paradox, and unique scenarios like Mozart's musical dice game. The text examines probability calculations in diverse contexts, including smartphone unlock patterns, board games like Nine Men's Morris, and gambling games like Win for Life and Keno. Each example illustrates how combinatorial methods can be used to calculate possible combinations and determine probabilities. The chapter analyses Key mathematical concepts through practical examples, such as smartphone unlock codes, Dyck walks, Catalan numbers, and Schröder-Hipparchus numbers. The chapter highlights how seemingly simple counting problems can reveal complex mathematical relationships and provide insights into randomness, pattern recognition, and probability estimation across different domains. The narrative emphasizes that combinatorics and probability are powerful tools for understanding uncertainty.

INTRODUCTION

When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth
Arthur Conan Doyle

Combinatorics and probability are two fundamental branches of mathematics. Although distinct, they are deeply intertwined in their application and study. Combinatorics examines the counting and arrangement of objects according to specific rules, while probability focuses on measuring uncertainty and predicting events. Together, these fields offer powerful tools for addressing various challenges.

A simple example can serve as an introduction: for each wheel of the lottery, five numbers are drawn from 90. How many different combinations can be drawn? What is the probability of guessing the five? Combinatorial calculus tells us that $C_{90,5}$ = 43,949,268 so the probability is 1/43,949,268, which is extremely low.

Wanting to summarize the possible combinations, these are the results:

Two numbers: 4,005 combinations
Three numbers: 117,480 combinations
Four numbers: 2,555,190 combinations
Five numbers: 43,949,268 combinations

Of course, there are many other versions of this game; the American Powerball lottery is based on the choice of six numbers: five between 1 and 69, one between 1 and 26. Consider another simple problem: We want to calculate the probability of getting exactly 3 "heads" by flipping a coin 5 times. Combinatorics provides us with the solution, the number of ways in which we can get 3 "heads" and 2 "tails" in 5 tosses is given by:

$$C_{5,3} = \frac{5!}{3!(5-3)!} = 10$$

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Let us calculate the probability of each combination. Since each outcome (heads or tails) has a probability of 1/2, the probability of getting exactly 3 "heads" is:

$$P(3) = C_{5,3}\left(\frac{1}{2}\right)^5 = \frac{5}{16}$$

Let us now examine some more or less well-known situations.

MONTY HALL PROBLEM

The first case considered, which is essentially related to probability, is known as the Monty Hall problem (Meylani, 2023), derived from a television game called "Let's Make a Deal" conducted by Monty Hall, pseudonym of Monte Halperin (1921-2017). Here is how it worked:

1. There are three doors. Behind one is a Machine, while behind the other two are Goats.
2. The contestant chooses a door, and the presenter, who knows what is behind each door, opens one of the other two doors, which has a goat behind it.
3. Monty then asks the contestant if he wants to change his initial choice to the other door.

The question is, should the competitor change his choice?

The Monty Hall problem can also be viewed in combinatorial terms because the contestant must consider all possible combinations of initial choices and intermediate outcomes. For example, there are six ways in which the prizes can be arranged behind the three doors distinguishing the two goats (MG_1G_2, MG_2G_1, G_1MG_2, G_2MG_1, G_1G_2M, G_2G_1M), and after the handler opens a door, some of these combinations are eliminated. All this can be represented as follows by supposing that the contestant chooses the second door and Monty Hall opens the door with a goat:

Table 1. Possible combinations and choices

MG_1G_2	**M**G_2G_1	G_1M**G2**	G_2M**G1**	G_1G_2**M**	G_2G_1**M**

One wins the car by changing the initial choice in two out of three cases (vos Savant, 1990).

PROBLEM OF BIRTHDAYS

An interesting example of combinatorics and probability is the birthday problem, which concerns the probability that at least two people share the exact date of birth in a randomly chosen group (DasGupta, 2005; Borja & Haigh, 2007). The paradox of the birthday was defined in 1939 by Richard von Mises (1883-1953).

For simplicity, ignore leap years and assume that every day of the year is equally likely as a date of birth. If we have only one person, there are 365 possible birthdays, so the probability that no one else has the same birthday is 1. If we add a second person, there are now 364 days remaining, so the probability that this second person will not share the first person's date of birth is 364/365. If we add a third person, the probability that they will not share the date of birth of the first two people is 363/365, and so on.

Thus, the probability p that *no one* in the group of n people shares a date of birth is:

$$p = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365 - n + 1}{365}$$