Chapter 3
From Ordinary Generating Functions to Combinatorial Calculus

ABSTRACT

The chapter explores fundamental concepts of combinatorics, focusing on ordinary generating functions (FGOs) and their applications in counting problems. FGOs are powerful tools for analyzing sequences by transforming them into algebraic expressions. They help in solving problems related to permutations, combinations, and arrangements. The chapter demonstrates how FGOs can generate sequences like Fibonacci numbers and solve real-world problems like binary sequence formation. The chapter then discusses different types of permutations, including simple permutations, circular permutations, and permutations with repetition. It also covers arrangements, which are ordered selections of elements and their variations with repetition. Combinations, where order does not matter, are analyzed, particularly in games like poker and probability calculations. Special cases include combinations with repetition and dismutations, a concept where no element remains in its original position.

INTRODUCTION

Ordinary generating functions are a natural language for combinatorics

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Ordinary generating functions (OGFs) are among the most powerful and flexible tools in combinatorics (Cameron, 1994; Lovász, 2007; Harris et al., 2009; Roberts & Tesman, 2009; Erickson, 2013; Mladenović et al., 2019; Bóna, 2023). They provide an elegant framework for analysing and manipulating numerical data, transforming complex counting problems (Sagan, 2020) into straightforward algebraic expressions. Practically speaking, an OGF associates a series of powers with a list of values, allowing their properties to be studied through algebraic techniques and providing a compact and unified way to handle an entire set of terms as a single object.

OGFs go far beyond solving isolated problems; they offer a broader perspective for exploring structurally similar families of combinatorial objects. Through their use, it becomes possible to uncover relationships between different configurations, develop advanced methods beyond elementary techniques, and apply insights to areas like number theory, probability, and the structural analysis of algorithms.

Let us consider the Fibonacci sequence: the number that grows by adding the previous ones: 1, 1, 2, 3, 5... Formally, it is defined by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$; the ordinary generating function is $G(x) = \sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$ it allows us to quickly calculate the terms of the sequence without having to resort to recursive methods.

This ordinary generating function (OGF) not only produces the Fibonacci numbers but also provides a versatile framework for tackling combinatorial problems. Consider the task of counting binary strings of length n that do not

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contain consecutive 1s. Such binary patterns are particularly relevant in computer science applications, such as data compression and encoding schemes.

Under the stated constraint, if the final digit is “0,” the preceding part can be any valid binary string of length $n-1$. If it ends in “1,” the digit before that must be “0,” which limits the preceding part to a valid configuration of length $n-2$. This recurrence mirrors the Fibonacci relation, establishing a direct link between the count of such valid binary strings and the Fibonacci numbers.

Ordinary generating functions help us to directly calculate sequence terms and provide a powerful methodology for solving combinatorial problems, making it easier to count configurations that would otherwise be complex to manage.

Permutations (Bóna, 2022), arrangements, combinations, and dismutations examined in this chapter are not directly ordinary generating functions (OGFs) but can be represented through them.

For example, the generating function for permutations is $P(x) = \sum_{n=0}^{\infty} n!x^n = \frac{1}{1-x}$.

To obtain the formula of simple permutations, *n!*, we can use the power series development of the ordinary generating function. The power series is $P(x) = 0!x^0 + 1!x^1 + 2!x^2 + 3!x^3 + \cdots = 1 + x + 2x^2 + 6x^3 + \ldots$ the constant term 1 represents the permutation of 0 elements, the subsequent terms represent those of 1, 2, 3, ... elements. The coefficient of x_n in this series is n! because each term $n!x_n$ represents the number of permutations of *n* objects. This follows from the very definition of the generating series, constructed in such a way that each term represents the number of permutations of *n* objects. Consequently, the number of permutations of *n* distinct objects is given by the coefficient of x_n in the power series associated with its ordinary generating function.

For combinations, the generating function is $C(x) = (1 + x)^n$. We expand the power series by means of the binomial theorem by obtaining $C(x) = \sum_{n=0}^{\infty} \binom{n}{k} x^n$. The term in the summation represents the number of combinations of *n* objects taken *k* at a time multiplied by the corresponding power. In this series, the coefficient of x^k is exactly $\binom{n}{k}$.

The same can be said for all the other formulas.

It should be emphasized that to enumerate permutations of a set with repeated elements, exponential generating functions must be used. In summary, ordinary and exponential-generating functions are valuable tools for solving combinatorial problems (Lamanna et al., 2022). The choice of the most suitable depends on the specific characteristics of the task at hand.

Having clarified these aspects, we can proceed to the examination of the fundamental relationships of combinatorial calculus.

SIMPLE PERMUTATIONS

Permutations refer to the order in which the elements of a set are arranged. For example, if we have a set of letters like {A, B, C, D}, there are 24 cases:

ABCD ABDC ACBD ACDB ADBC ADCB BACD BADC BCAD BCDA BDAC BDCA

CABD CADB CBAD CBDA CDAB CDBA DABC DACB DBAC DBCA DCAB DCBA

In a permutation, each element of the set appears only once, and the order is important.

To better understand the meaning of the term, we can refer to a famous photo in which eleven workers can be seen eating lunch, sitting on a beam suspended 250 meters high.