Chapter 1
Introduction to Combinatorics

ABSTRACT

Combinatorics, a branch of mathematics that studies the possibilities of selecting, arranging, and grouping elements of a set, has ancient roots in various cultures. The chapter traces the history of combinatorics, from its origins in the ancient East to modern developments. Key figures such as Pascal, Leibniz and Fibonacci are highlighted as important contributions. The chapter illustrates the connection of combinatorics with graph theory and other complex concepts thanks to figures like Paul Erdős and Ronald Graham. The most recent developments in combinatorics are examined, focusing on the impact of artificial intelligence and machine learning. It highlighted how these technologies open new avenues for optimizing graph problems. Combinatorics is fundamental for game theory, circuit design, social networks, and drug discovery. The chapter introduces theorems such as those of Szemerédi and Green-Tao that have brought new points of view. Finally, it presents several simulations to clarify better concepts, such as the drunken path and the Sierpinsky triangle.

A BRIEF HISTORY OF COMBINATORICS

Not knowing what happened in the past would be like being a child forever

Cicerone

Combinatorics, a branch of mathematics dedicated to the study of the possibilities of selecting, arranging, and grouping the elements of a set, has a long history with roots in various cultures and historical periods (Wilson & Watkins, 2013). The I Ching, or Book of Changes, is an ancient Chinese text (thought to date back to at least 1000 BC) with deep connections to combinatorics principles (Redmond & Hon, 2014). It is based on 64 hexagrams, each consisting of six lines. Each line can be yang (whole) or yin (broken), representing a primitive binary system. The basic eight trigrams (combinations of 3 lines) generate the 64 hexagrams (6 lines). It represents one of the first systematic applications of the combinatorial principle: $2^6 = 64$. The order of the lines is significant, introducing the concept of permutations. There are 8! (40,320) possible ordering of trigrams, even if only a few are used. The I Ching represents one of the earliest examples of systematic combinatorial thinking, serving as a bridge between mathematics and divination. This text profoundly influenced Chinese thought, particularly in Confucianism, Taoism, and Western thought, with philosophers such as Leibniz and Jung being inspired by it.

The most famous example of a combinatorial arrangement is the magic square known as Lo Shu, with a written trace starting from 650 BC. According to legend, the Lo River was in flood, so a turtle with a design engraved on its back emerged to save the population from the flood. The pattern was a 3x3 grid, each containing one of the numbers between 1 and 9. Each row, each column and both diagonals of the square gave a sum of 15. This poses the problem of how to arrange the figures to achieve this result.

We find traces of combinatorial reasoning in Chinese, Babylonian, Egyptian and Indian civilizations. The Indian mathematician Pingala, in his text "Chandaḥśāstra", written in the fourth century BC, presented a method of combinatorial enumeration (Berstel & Perrin, 2007). He examined ways of combining syllables of different lengths (Short and

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Long) to produce poetic meters. For example, with three syllables, one can get eight different rhythms: BBB, LLL, BLL, LBL, LLB, BLB, BBL.

While combinatorics wasn't a primary focus for the Greeks and Romans, we do find intriguing hints of its underlying principles in their work. Take Zenodorus, for instance, and his study of the isoperimetric problem. In the language of combinatorial optimisation, this challenge asks: Which configuration maximises a specific property (in Zenodorus's case, the area) while keeping a constraint (the perimeter) constant? This line of reasoning, though not directly employing formal combinatorial methods, shares a conceptual kinship. It involves evaluating multiple possibilities to pinpoint the one that optimises a given characteristic. This approach strongly foreshadows modern decision theory and operations research, disciplines that now extensively leverage combinatorics to solve complex optimisation challenges (Acerbi, 2003).

Another scholar to consider was Hipparchus; although he did not directly contribute to combinatorics, his systematic work on classification in astronomy can be seen as an embryonic form of combinatorial thinking (Acerbi, 2003).

Pappus of Alexandria, who lived between the third and fourth centuries AD, was one of the most important mathematicians of the Hellenistic era. In his "Mathematical Collection" work, Pappo explores various geometric and mathematical topics. Although he does not deal directly with combinatorics, his approach to some geometric issues includes aspects that we can consider anticipatory of enumerative combinatorics. Enumerative combinatorics is concerned with calculating the number of ways in which certain objects can be organized or counted.

For example, one of the challenges studied by Pappus concerns the calculation of circle configurations within a given geometric figure. This type of question requires identifying possible arrangements of elements (in this case, circles) that satisfy a set of conditions or constraints. One example of this geometric exploration involves tangents, where Pappus asks how many circles can be positioned to be tangent to two parallel lines and another circle. The task involves determining all possible ways to arrange the circles while respecting the given geometric conditions—an approach that, in a sense, anticipates combinatorial analysis in finding solutions within a constrained set of possibilities.

Although formalized combinatorics as we understand it today did not exist, these scholars laid the foundations for future exploration of these mathematical concepts.

One of the most illustrious mathematicians of medieval India was Bhāskara II (1114–1185). His main work, the Līlāvatī, encompasses a wide range of mathematical inquiries, covering various topics, including algebra, geometry, and, notably, combinatorics (Biggs, 1979). Bhāskara II studied arrangements and selections of elements, laying the groundwork for a more systematic understanding of these techniques. One of his contributions to combinatorial mathematics is the analysis of permutations and combinations. He introduced the concept of bhāgas, or groupings of elements appearing in a particular order. Bhāskara explains, for example, the method for calculating the number of permutations of n elements, which is reflected in the formula we know today as $n!$

The use of factorial formulas (n! being the product of all positive integers less than or equal to n) appears in his discussion of how many ways Vishnu could hold his four characteristic objects—a shell, a disc, a mace, and a lotus flower—in his four hands. He also explores advanced combinatorial techniques, such as determining the number of ways letters in a word can be arranged and selecting specific values from a set. In this, he demonstrates a mastery of combinatorial reasoning and a broader vision of mathematics as a tool for solving practical questions.

The ancient mathematicians and scientists of these civilizations faced concrete challenges, such as organizing crops, carrying out commercial transactions, and solving problems related to the calendar. The solutions to these problems required combinatorial skills, even if, seen today, they seem quite rudimentary. In reality, exciting signals will be taken up in other eras. However, we should not forget that the first mathematical bases that will be found in the most modern developments are also beginning to appear.

Even simpler puzzles from later periods can be viewed through a combinatorial lens. Consider Alcuin of York's (735–804) classic "chickens and rabbits" problem from the ninth century. It asks one to determine the count of rabbits and chickens given the total number of heads and legs. While commonly solved using a system of linear equations (e.g., with 50 heads and 140 legs, the owner has 30 chickens and 20 rabbits), it also has a clear combinatorial interpretation.

At its core, it is a counting exercise—determining how we can assign a set of features (heads and legs) to different categories (chickens and rabbits). One might consider all possible distributions of animals, such as 50 chickens and no rabbits, or vice versa. As can be quickly verified, there are 51 such combinations, but only one satisfies all given