Chapter 7
Differential Calculus of Multivariable Functions

ABSTRACT

This chapter focuses on differential calculus of multivariable functions. We generalize the definition of differentiation to such functions. The limit and continuity in a point on a plane will be defined as we had in Chapter 4. The rules of differentiations of the functions which are introduced in Chapter 5, can also be used for multivariable functions with the difference that we call them partial derivatives. After definition of degree for homogeneity the Euler's Theorem we found a relation between partial derivatives of a multi-valued function and degree of homogeneity. Several applications of partial differentiation will be given through this chapter. We mention that the Taylor's formula for single-valued function can also be generalized to multi-valued functions.

7.1 INTRODUCTION

We know that the area of a rectangle is obtained from the relation $S=ab$, where a and b are the length and width of the rectangle, respectively. To calculate the volume of a rectangular cube, we use the relationship $V=abc$, where a and b are the lengths of the base rectangle, and c is the width of the rectangular cube. The first relation shows that the value of the area depends on a and b. We can consider it as a two-variable function. In the second relation, assuming, for example, $a=x_1$, $b=x_2$ and $c=x_3$, it shows the volume in terms of functions of x_1,x_2 and x_3, i.e. $f(x_1,x_2,x_3)$. These two examples show that in many problems we are dealing with multivariable functions. The function of the total cost of each family depends on incidental expenses

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such as clothing, food, health care, etc. They are commonly used in management for various purposes, such as optimization, forecasting and decision-making.

In general, a function can be with n variables $x_1, x_2, \ldots, x_n$, but in this chapter, we focus on functions with two or three variables.

7.2 MULTIVARIABLE FUNCTIONS

If we write the function f with the rule $w=f(x_1, x_2, \ldots, x_n)$, then we have a multivariable function with a set of ordered pairs $((x_1, x_2, \ldots, x_n)w)$ that, like a single function, none of the two different ordered pairs have equal members. With this argument, an n-variable function maps the n-dimensional real space, denoted by $\mathbb{R}^n$, to the 1-dimensional real space $\mathbb{R}$, that is, $f{:}\mathbb{R}^n \rightarrow \mathbb{R}$.

The largest subset of the set $\{(x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}\}$ in which the function f is defined, is called the domain of the function and the set of values that w can take is called the range of the function denoted by D_f and R_f, respectively. The examples we examine in this chapter are mostly for the functions of two variables $z=f(x,y)$ or the functions of three variables, $w=f(x,y,z)$, whose results can be generalized for n-variables functions. Before we go further, let's have a picture about some specific graphs of $f(x,y,z)=0$.

In Chapter 1, we mentioned that a general form for a second-order surface given by

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where A,B,C,D,E,F,G,H,I and J are real constant coefficients. The graph of any such equation is called *quadric surface*. Since the coefficients of this equation determine the shape and orientation of the quadric surface in space, hence for the time being we assume $D=E=F=0$. That is, we have the following equation.

$$Ax^2 + By^2 + Cz^2 + Gx + Hy + Iz + J = 0$$

According to this equation, we introduce some types of surfaces that are more well-known forms.

1. If $A=B=C$, then the quadric surface is called a *sphere* with radius r and center (x_0, y_0, z_0) and will be given by

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

If $x_0=y_0=z_0=0$, then the center of sphere is located on origin (Figure 1).