Chapter 4
Limits and Continuity

ABSTRACT

This chapter focuses on limits and continuity at a point and on an interval. Infinite limits and the limits at infinity are considered then, concepts of the asymptotes will be declared. Concepts of right and left limits and right and left continuity will be defined. Continuity on an interval will be used to introduce the intermediate value theorem which is also known as Bolzano's theorem. By this theorem we find the interval where the root of an equation may exist. To complete this chapter, we add a section namely Sequences and Series which is optional. After defining the concepts of convergent and divergent for sequences and series we introduce some tests for convergent and divergent.

4.1 INTRODUCTION

Let's consider the function $f(x) = \frac{x^2 - 5x + 6}{x - 2}$. Clearly, $D_f = \mathbb{R} - \{2\}$. In other words, the function is not defined in x=2. What is the value of $f(x)$ when x=2? How do the values of $f(x)$ behave when x is near but different from 2?

We want to consider the behavior of the function near 2. For this purpose, we consider the following table:

Table 1.

x	1 1.5 1.9 1.99 1.999 … 2 … 2.001 2.01 2.1 2.5 3
$f(x)$	-2 -1.5 -1.1 -1.01 -1.001 ? -0.999 -.99 -0.9 -0.5 0

We observe that as x approaches 2, the value of $f(x)$ approaches -1. More precisely, the value of $f(x)$ can be as close to -1 as we want, provided that the value of x is close to 2. That is, we can reduce the difference between $f(x)$ and -1 as much as

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we want, provided that the difference between x and 2 becomes smaller. Because in the table one of the differences is positive and one is negative, it is better to use the symbol of absolute value.

Thus, if $|x–2|$ is chosen small enough, then $|f(x)+1|$ can be chosen as small as desired. If we use two positive numbers ε and δ for these two differences, we can find some positive number δ such that $|f(x) + 1| < \delta$, whenever $|x - 2| < \varepsilon$.

It should be noted that to define the limit of the function f at a point a, the function f does not need to be defined at this point. Even if the function is meaningful at this point, the limit of the function at this point may not be equal to the value of the function at this point.

Now, according to these contents, we can define the limits of the function and express theorems related to them.

4.2 DEFINITIONS AND THEOREMS OF LIMITS

We assume that the function f is defined near the point a. We say that the limit of the function f when x tends to a is equal to L and we write

$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that whenever $0 < x - a < \delta$, we have $|f(x) - L| < \varepsilon$.

We call L the *limit* of the function, which must be unique if it exists.

Before continuing to examine the limit problems, we state the theorems related to the limits.

Theorem 1. If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then

(i) $\lim_{x\to a} [f(x) \pm g(x)] = L_1 \pm L_2$
(ii) $\lim_{x\to a} [f(x) . g(x)] = L_1 . L_2$
(iii) $\lim_{x\to a} [f(x)/g(x)] = L_1/L_2$
(iv) $\lim_{x\to a} [\circledast f(x)] = \circledast f(L)$

where $\circledast$ represents an operator such as a radical, logarithm or trigonometric operators.

Example 1: Prove that, $\lim_{x\to 1}(8x - 5) = 3$.

Solution: We must show that L=3. For this purpose, we consider $|f(x)–L|$.

We must have

$$|f(x) - \mathrm{L}| = |8x - 5 - 3| = 8|x - 1| < \varepsilon \Rightarrow |x - 1| < \varepsilon/8.$$

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