Chapter 3
A Discussion on Non-Convex Optimization Problems Arising in Supply Chain Design and Finance

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ABSTRACT

Non-convex optimization problems belong to a class of classical nonlinear optimization problems, which are often difficult to solve. An optimization problem becomes non-convex due to the presence of non-convex functions in the objective function or constraints. A function is a convex function if its Hessian matrix is positive and semi-definite for all values; otherwise, it is a non-convex function. A Hessian matrix is called positive semi-definite when the eigenvalues of the matrix are non-negative. A non-convex function can be either a concave function or a function that is neither a concave nor a convex function. A concave function is always negative semi-definite, indicating that the eigenvalues of the matrix are non-positive. This chapter starts with a short introduction to non-convex problems, followed by a discussion on different non-convex problems arising in supply chain and finance. Thereafter, the authors discuss different algorithms used for solving non-convex problems. Finally, the chapter conclude with the limitations of different algorithms.

INTRODUCTION

Non-convex optimization problems belong to a class of classical nonlinear optimization problems, which are often difficult to solve. Non-convex problems have been of interest for the last eighty years. An optimization problem becomes non-convex due to the presence of non-convex functions in the objective function or constraints. A non-convex function can be either a concave function or a function that is neither concave nor convex. Let's consider a non-convex optimization problem of the following kind:

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$$\min_{x} f\left(x\right)+\phi\left(x\right) \tag{1.1}$$

Subject to $g_i(x)\leq 0$, $\forall i=1,2,\ldots,I$ (1.2)

$$x_k^l \leq x_k \leq x_k^u \forall\, k=1,\ldots,n\ . \tag{1.3}$$

where $g_i(x)$ and $f(x)$ are convex functions, ϕ(x) is either a non-convex function. Constraints (1.3) set the bounds for the variable x$k_,$ which may be continuous or discrete. Next, we will briefly discuss the properties of non-convex functions.

NON-CONVEX FUNCTION

A function is non-convex when the function does not act as a convex function. A function is a convex function if and only if its Hessian matrix is positive semi-definite for all values; otherwise, it is a non-convex function. A hessian matrix is called positive semi-definite if and only if all eigenvalues of the matrix are non-negative. For example, consider a function of n variables $f(x)$ defined on a convex set D. $f(x)$ is said to be a convex function when the following conditions hold.

- For any two points x^1 and x^2 and $0\leq\lambda\leq 1, f\left[x^1\lambda+x^2\left(1-\lambda\right)\right]\leq f\left(x^1\right)\lambda+f\left(x^2\right)\left(1-\lambda\right)$.
- The hessian matrix of $f(x)$ is positive semi-definite for all values of $x_1,\ldots,x_n$.

Figure 1a shows an example of convex functions. A non-convex function can be either a concave function or a function that is neither a concave nor a convex function. A function is a concave function if its hessian matrix is negative semi-definite, which happens only when all eigenvalues of the matrix are non-positive. For example, consider a concave function of n variables ϕ(x) defined on a convex set D. ϕ(x) is said to be concave when the following conditions hold.

- Foranytwopointsx*1*andx*2*and0≤λ≤1, $0\leq\lambda\leq 1, f\left[x^1\lambda+x^2\left(1-\lambda\right)\right]\geq f\left(x^1\right)\lambda+f\left(x^2\right)\left(1-\lambda\right)$.
- The hessian matrix of $f(x)$ is negative semi-definite for all values of $x_1,\ldots,x_n$.

An example of a non-convex function, which is neither concave nor convex, includes an inverse S-shaped function. An inverse S-shaped function is concave in nature initially up to a certain point above which the function changes its shape to convex. The point is often referred as a deflection point or economic point. In Figure 1.1b, we show an inverse S-shaped function. Historically, non-convex optimization problems have been of interest due to several real-world applications. Next, we briefly describe some of the non-convex application problems.