Chapter 14
Riesz Potential With Logarithmic Kernel in Generalized Hölder Spaces:
Theorems on Inversion and Isomorphisms

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ABSTRACT

The multidimensional Riesz potential type operators are of interest within mathematical modelling in economics, mathematical physics, and other, both theoretical and applied, disciplines as they play a significant role for analysis on fractal sets. Approaches of operator theory are relevant to researching various equations, which are widespread in financial analysis. In this chapter, integral equations with potential type operators are considered for functions from generalized Hölder spaces, which provide content terminology for formalizing the concept of smoothness, briefly described in the presented chapter. Results on potentials defined on the unit sphere are described for convenience of the analysis. An inverse operator for the Riesz potential with a logarithmic kernel is carried out, and the isomorphisms between generalized Hölder spaces are proven.

INTRODUCTION

Let a natural $n \geq 2$ stand for the dimension of R^n – the set of all vectors $x = \left(x_1, x_2, \ldots, x_n\right)$ with real coordinates, thus $x_i \in \mathrm{R}$, $i = \overline{1, n}$. Being supplied with the metric

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$$|x-\sigma| = \sqrt{(x_1-\sigma_1)^2+\ldots+(x_n-\sigma_n)^2}\ ,$$

R^n is a Euclidean space, and $|x|=|x-0|$, while an inner product in R^n could be defined as

$$x\cdot\sigma = x_1\sigma_1+\ldots+x_n\sigma_n\,,$$

or, due to the parallelogram identity, $x\cdot\sigma = \frac{1}{2}\left(|x+\sigma|^2-|x|^2-|\sigma|^2\right)$. This allows to express the metric through the inner product:

$$|x-\sigma| = \sqrt{|x|^2+|\sigma|^2-2\,x\cdot\sigma}$$

A unit sphere S^{n-1} in R^n is considered as a set of points, equidistant from the origin by a distance of one, i.e.

$$S^{n-1} := \left\{\sigma\in R^n : |\sigma|=1\right\}.$$

It is clear, that, for an arbitrary $x\in R^n$, $x/|x|\in S^{n-1}$. For $x,\sigma\in S^{n-1}$, the relation (1) becomes

$$|x-\sigma| = \sqrt{2}\sqrt{1-x\cdot\sigma}$$

The purpose of the paper is to investigate solutions of an integral equation with a spherical convolution operator

$$(K\,f)(x) = \int_{S^{n-1}} k(x\cdot\sigma)\,f(\sigma)\,d\sigma,\ x\in S^{n-1},$$

which is to be achieved through applying the Fourier—Laplace multiplier theory. In a nutshell, the latter assume, that if the preimage *f* can be decomposed into a Fourier series, then the coefficients of such a decomposition for the image *Kf* are defined in the specific way, briefly described below.

Eventually, the approach allows to achieve theorems on reflections between various function spaces and outline the properties of integral equations. In the present paper, isomorphisms between generalized Hölder spaces are proved for a kind of *Kf* – the Riesz potential type operator with a logarithmic kernel

$$\left(K^{\alpha}\,f\right)(x) = \int_{S^{n-1}} \frac{f(\sigma)}{|x-\sigma|^{n-1-\alpha}}\ \ln\frac{r}{|x-\sigma|}\,d\sigma,\ x\in S^{n-1},$$

where $r\in R$ is a fixed constant. Due to the relation (2), K^{α} is indeed a particular instance of *K* with