Mathematical Knowledge Management

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INTRODUCTION

Mathematical knowledge is significantly different from other kinds of knowledge. It is abstract, universal, highly structured, extraordinarily interconnected, and of immense size. Managing it is difficult and requires special techniques and tools.

Mathematicians have developed (over the last two or three millennia) many techniques for managing mathematical knowledge. For example, there is a large collection of techniques based on the use of special symbols and notations. Although these techniques are quite effective and have greatly advanced mathematical practice, they are primitive in the sense that the only tools they require are pencil and paper, typesetting machines, and printing presses.

Today mathematics is in a state of transition. Mathematicians are using the Internet in new ways to find information and to share results. Engineers and scientists are producing new kinds of mathematical knowledge that is oriented much more to practical concerns than to theoretical interests. This is particularly true in the field of software development where software specifications and code are forms of mathematical knowledge. Computers are being pushed to perform more sophisticated computations and to mechanize mathematical reasoning. Mathematical knowledge, as a result, is being produced and applied at an unprecedented rate.

It is becoming increasingly difficult to effectively disseminate mathematical knowledge, and to ascertain what mathematical results are known and how they are related to each other. Traditional ways of managing mathematical knowledge are no longer adequate, and current computer and communication technology do not offer an immediate solution. Since mathematical knowledge is vital to science and technology, and science and technology is vital to our society, new ways of managing mathematical knowledge based on new technology and new theory are needed.

This article introduces the main issues of managing mathematical knowledge. It is organized as follows. The Background section describes mathematics as a process of creating, exploring, and connecting mathematical models. The special characteristics of mathematical knowledge and the four main activities that constitute the management of mathematical knowledge are discussed in the Main Focus of the Article. The Future Trends section introduces *Mathematical Knowledge Management* (MKM), a new field of research, and discusses some of the challenges it faces. The article ends with a conclusion, references, and a list of key terms.

The management of mathematical knowledge is an emerging field of research. Researchers are just starting to build a foundation for it. This article focuses on the core concerns of the field. Except for a few remarks, it does not discuss the parallels between mathematical knowledge management and mainstream knowledge management. Nor does it discuss how techniques for managing mathematical knowledge can be applied to the management of other kinds of knowledge. These are important topics for future research.

BACKGROUND

People often associate mathematics with a body of knowledge about such things as numbers, spatial relationships, and abstract structures. However, this view of mathematics is misleading. It suggests that mathematics is something static and dead, but mathematics is actually the opposite—dynamic and alive. It is more productive and accurate to view mathematics as a process for comprehending the world that consists of three intertwined activities (Farmer & von Mohrenschildt, 2003).

The first activity is the *creation of mathematical models* that represent mathematical aspects of the world. Mathematical models come in many forms. A well-known and important example is the model of real number arithmetic composed of the set of real numbers, and operations and relations involving the real numbers such as $+$, $\times$, and $<$. Real number arithmetic includes various *submodels* such as arithmetic of the natural numbers 0,1,2,... and arithmetic of the rational numbers like 2/3, 31/17, and so forth. Real number arithmetic and its submodels capture the essential elements of counting, measurement, motion, and much more. Real number arithmetic itself is a submodel of complex number arithmetic and many other mathematical models.

The second activity is the *exploration of mathematical models* to learn what they say about the mathematical aspects of the world they model. There are several means of exploration. The explorer can state a conjecture about a model and then attempt to *prove* that the conjecture is true by virtue of being a logical consequence of the defining properties of the model. The explorer can also formulate a problem concerning the model and then *compute* a solution to it by mechanically manipulating a representation of the problem using rules determined by the model. A third approach, which is sometimes very effective, is to *visualize* some facet of the model with a diagram, picture, or animation.

The last activity is the *connection of mathematical models* by identifying and recording relationships between models. Models can be related to one another in various ways. Examples includes two models being equivalent in a certain sense, one model containing another as a submodel, and one model generalizing another model. A collection of interconnected models facilitates the creation and exploration of new models. New models can be built from old models, and then the results about the old models can be applied to these new models according to how they are connected. Thus, models rarely need to be developed from scratch.

MAIN FOCUS OF THE ARTICLE

Mathematical knowledge is knowledge about mathematical models. Each piece of mathematical knowledge is understood relative to a *context* of a mathematical model or group of mathematical models. For example, the statement "there is no square root of -1" is true in the model of real number arithmetic, but actually false in complex number arithmetic (the square root of -1 is the complex number *i*). Although a piece of mathematical knowledge is not meaningful without its context, the context of mathematical knowledge is often not explicitly stated. For example, one might say that as a mathematical fact, "every nonzero number has a multiplicative inverse" without mentioning the context of the statement. Of course, this statement is true for rational number arithmetic and real number arithmetic, but false for natural number arithmetic.

The context for understanding mathematical knowledge is analogous to the context for understanding other kinds of knowledge. Knowledge, mathematical or otherwise, that is applied out of its proper context is not reliable. The context of a piece of knowledge, mathematical or otherwise, is often imprecise or not fully articulated. However, a context for mathematical knowledge, unlike a context for many other kinds of knowledge, can be made as precise as is desired.

Mathematical knowledge is *direct* knowledge about mathematical models, but it is also *indirect* knowledge about the mathematical aspects of the world which are being modeled. As indirect knowledge, mathematical knowledge is useful, often even vital, to engineers and scientists. It is routinely used to help solve real-world problems.

Mathematical knowledge has several characteristics that sharply distinguish it from other kinds of knowledge. These characteristics make managing mathematical knowledge significantly different from managing other kinds of knowledge.

Abstractness

A mathematical model is an abstraction of the world; it ignores everything about the world except some part of the world's underlying mathematical structure. Other kinds of knowledge can be abstract, but mathematical knowledge is inherently abstract. Moreover, mathematics is, to a large degree, the study of abstractions.

Universality

Direct knowledge about a mathematical model is indirect knowledge about any situation in the world that exhibits the mathematical structure captured by the model. For example, it is true in the model of rational number arithmetic that, for any two integers m,n, if m/n is an integer, then $m = m/n + \ldots + m/n$ (n times) is sum of equal integers. As a result, *any* set of m objects can be divided into n subsets of equal size if m is divisible by n. Mathematical knowledge is thus universal in the sense that it can be applied to every domain of interest that exhibits the right kind of mathematical structure.

Language

Mathematical knowledge is usually expressed in a language with a carefully controlled syntax and a precise, unambiguous semantics. The language allows one to express statements about a certain collection of objects. The language may be an *informal* language based on a natural language such as English in which ordinary words such as "implies" and "function" have special meanings. The language may also be a *formal* language that can be read, analyzed, and presented by software.

Semantics

Unlike other kinds of knowledge, mathematical knowledge can be given a precise semantics. This is usually done by representing the context of the mathematical knowledge as a "mathematical theory." For example, an *axiomatic theory* is a pair $T = (L,A)$ where L is a language and A is a set of statements of L called *axioms*. The axioms express properties that the objects of L are assumed to possess. A mathematical model is a *model* of T if it has the