Chapter 15
Price Systems for Random Amounts:
A Unified Approach

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ABSTRACT

Internal consistency—or "coherence"—of a price system is the basis of several key concepts in many fields, such as subjective probability (in Probability Theory), no-arbitrage pricing, and risk measures (in Mathematical Finance). Furthermore, Actuarial Mathematics uses coherence to describe the analytical form of risk premia, and an analogous approach has recently been proposed for firms' valuation. Technically, it amounts to a characterisation of functionals with particular properties (a typical goal in Functional Analysis), which translates into a numerical representation of preferences along the traditional guidelines of Decision Theory, whose analogies with Mathematical Finance are numerous and really impressive. This is explored in this chapter.

INTRODUCTION

The internal coherence of a price system for random amounts is a cornerstone common to various disciplines. In *Probability Theory* it is the basis for the definition of *subjective probability*. In *Mathematical Finance* it represents the unavoidable tool for the *no-arbitrage valuation* of financial assets traded in a market. Moreover, it is used to define *risk measures*, since they are intended as prices, expressed by an opportune market, in order to acquire a coverage for the risk positions to be measured. In the last years, also *Actuarial Mathematics* has been using this approach to describe the analytical form of risk premia. Recently, an analogous approach has been proposed for *firms valuation* as well.

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Technically the language and many (classical and not) results derive from Functional Analysis and allow to characterize classes of functionals with opportune properties. This process is developed along the traditional lines of Decision Theory which, starting from a set of axioms on the preferences of a decision maker, tries to find functionals representing said preferences. Very often the analogies of the approach and the results of Decision Theory and Mathematical Finance are impressive.

The first section of the paper shows how price systems in financial markets can be represented through linear, sublinear, convex and star-shaped functionals. The second section is dedicated to coherent, convex and star-shaped risk measures. The third section concerns insurance premia, and the last section is devoted to firms valuation. Some considerations about arbitrage, prices and representation functionals are relegated in the Appendix.

BACKGROUND

In Probability Theory, the definition of a subjective probability was first given by de Finetti (1931b, 1937, 1949), by making use of the concept of absence of sure loss in a system of bets on a given set of events, or random variables. Such an approach is equivalent to impose absence of arbitrage for a price system of random amounts. Several disciplines are concerned with the problem of assigning a numerical value, or a "valuation", to random variables.

The main field of study is of course *Mathematical Finance*, where *asset pricing* problem is studied since the pioneeristic works of Harrison and Kreps (1979) and Harrison and Pliska (1981). Several studies focused on how to model a financial market and analyse how prices are formed: see., *e.g.*, classical Finance textbooks such as Dothan (1990), Duffie (2001), Hull (2014), and Pliska (1997). The basic idea is that, whenever a portfolio of traded assets can be built so as to perfectly replicate (*hedge*) the payoff of another asset, the latter asset and the portfolio need to have the same price to avoid *arbitrages*, *i.e.*, sure gains without risk. The central role of arbitrage is since then emphasised in several cornerstone textbooks of the discipline, such as Björk (1999) and Karatzas and Shreve (1998), and applies to stock markets as well as bond ones (see., *e.g.*, Brigo & Mercurio, 2006); notably enough, it was already intuitively anticipated several years before by de Finetti (1935). It thus becomes crucial to investigate the properties of such a price functional, and to provide financial interpretations of them: several papers are devoted to study derivative securities (Black & Scholes, 1973; Merton, 1973), interest rates and riskless bonds (Heath, Jarrow, & Morton, 1992), transaction costs (Cuevas Dermody & Prisman, 1993), limitations to arbitrage (Pham & Touzi, 1999; Castagnoli & Favero, 2005; Castagnoli, Favero, & Tebaldi, 2011; Liu & Longstaff, 2004), market liquidity (Astic & Touzi, 2007; Pennanen, 2011), Choquet expectations (Chateauneuf, Kast, & Lapied, 1996), market completeness (El Karoui & Quenez, 1995), margins, dividends and so on; an impressive number of papers, furthermore, investigates prices in real markets in order to check whether they follow the "theoretical prescriptions" or violate them. In the beginning, *linear* (*i.e.*, homogeneous and additive) price functionals where used, but soon *sublinear* (*positively* homogeneous and *sub*additive) ones were studied, either because they conveniently take into account proportional "frictions" (Jouini & Kallal, 1995; Koehl, Pham, & Touzi, 1999), or because they are the natural result of pricing by super-hedging in incomplete markets (El Karoui & Quenez, 1995; Koehl, Pham, & Touzi, 2002; Aliprantis, Polyrakis, & Tourky, 2002). Soon it became clear that losing linearity had a deep impact even on the definition of market effectiveness: namely, arbitrages appear not to be the main concern, whereas the "real" effectiveness condition for well-behaved markets should be absence of