Uncertainty Operators in a Many-Valued Logic

U

Herman Akdag
University Paris6, France

Isis Truck
University Paris8, France

INTRODUCTION

This article investigates different tools for knowledge representation and modelling in decision making problems. In this variety of AI systems the experts' knowledge is often heterogeneous, that is, expressed in many forms: numerical, interval-valued, symbolic, linguistic, etc. Linguistic concepts (adverbs, sentences, sets of words…) are sometimes more efficient in many expertise domains rather than precise, interval-valued or fuzzy numbers. In these cases, the nature of the information is qualitative and the use of such concepts is appropriate and usual. Indeed, in the case of fuzzy logic for example, data are represented through fuzzy functions that allow an infinite number of truth values between 0 and 1. Instead, it can be more appropriate to use a finite number of qualitative symbols because, among other reasons, any arbitrary fuzzification becomes useless; because an approximation will be needed at the end anyway; etc. A deep study has been recently carried out about this subject in (Gottwald, 2007).

In this article we propose a survey of different tools manipulating these symbols as well as human reasoning handles with natural linguistic statements. In order to imitate or automatize expert reasoning, it is necessary to study the representation and handling of discrete and linguistic data (Truck & Akdag, 2005; Truck & Akdag, 2006). One representation is the *many-valued logic* framework in which this article is situated.

The many-valued logic, which is a generalization of classical boolean logic, introduces truth degrees which are intermediate between *true* and *false* and enables the partial truth notion representation. There are several many-valued logic systems (Lukasiewicz's, Gödel's, etc.) comprising finite-valued or infinite-valued sets of truth degrees. The system addressed in this article is specified by the use of $\mathcal{L}_M = \{\tau_0,...,\tau_i,...,\tau_{M-1}\}$ [1] a totally ordered finite set[2] of truth-degrees ($\tau_i \leq \tau_j \Leftrightarrow i \leq j$) between τ_0 (false) and τ_{M-1} (true), given the operators $\vee$ (max), $\wedge$ (min) and $\neg$ (negation or symbolic complementation, with $\neg\tau_j = \tau_{M-j-1}$) and the following Lukasiewicz implication $\rightarrow_L$: $\tau_i \rightarrow_L \tau_j = \min(\tau_{M-1}, \tau_{M-1-(i-j)})$

These degrees can be seen as membership degrees: x partially belongs to a multiset[3] A with a degree τ_i if and only if $x \in_{\tau_i} A$. The many-valued logic presented here deals with linguistic statements of the following form: x is v_α A where x is a variable, v_α a scalar adverb (such as "very", "more or less", etc.) and A a gradable linguistic predicate (such as "tall", "hot", "young"...). The predicate A is satisfiable to a certain degree expressed through the scalar adverb v_α. The following interpretation has been proposed (Akdag, De Glas & Pacholczyk, 1992):

$$x \text{ is } v_\alpha A \Leftrightarrow \text{“}x \text{ is } A\text{” is } \tau_\alpha - \text{true}$$

Qualitative degrees constitute a good way to represent uncertain and not quantified knowledge, indeed they can be associated with Zadeh's linguistic variables (Zadeh, 2004) that model approximate reasoning well. Using this framework, several qualitative approaches for uncertainty representation have been presented in the literature. For example, in (Darwiche & Ginsberg, 1992; Seridi & Akdag, 2001) the researchers want to find a model which simulates cognitive activities, such as the management of uncertain statements of natural language that are defined in a finite totally ordered set of symbolic values. The approach consists in representing and exploiting the uncertainty by qualitative degrees, as probabilities do with numerical values. In order to manipulate these symbolic values, four elementary operators are outlined: multiplication, addition, subtraction and division (Seridi & Akdag, 2001). Then two other kinds of operators are given: modification tools based on scales and symbolic aggregators.

BACKGROUND

Extending the work of Akdag, De Glas & Pacholczyk on the representation of uncertainty via the many-valued logic (Akdag, De Glas & Pacholczyk, 1992), several studies have been led where an axiomatic system for symbolic probability theory has been proposed (Seridi & Akdag, 2001; Khayata, Pacholczyk & Garcia, 2002).

The qualitative uncertainty theory we present here takes place between the classical probability theory and possibility theory. The idea is to translate the four basic operations respecting required properties with well-chosen formulas. The qualitative and the numerical models are linked together using the four symbolic operators that compute the qualitative operations.

These works related to the qualitative uncertainty theory have in common the following points:

- They are based on the association "probability degrees / logic": In their work, laws of probabilities are obtained thanks to logical operators.
- They have developed an axiomatic theory, which allows obtaining results either from axioms, or from theorems.
- In addition, they permit the use of uncertainty (and imprecision) expressed in the qualitative form.

The considered qualitative degrees of uncertainty belong to the graduated scale $\mathcal{L}_M$. The first step is to introduce a total order in the scale of degrees. Then in order to be able to translate in symbolic the different axioms and theorems of the classical probability theory, three elementary operators must be defined as in (Darwiche & Ginsberg, 1992). Indeed a symbolic addition (or a symbolic t-conorm, to generalize the addition), a symbolic multiplication (or a symbolic t-norm in order to be able to translate the disconditioning and the independence) and a symbolic division (to translate the conditioning) must be provided.

Moreover, another constraint is introduced: "*it is necessary that if C is the result of the division of A by B then B multiplied by C gives A*" (relationship between the qualitative multiplication and the qualitative division). This intuitive constraint has been proposed for the first time in (Seridi & Akdag, 2001). Another originality lies in the definition of a qualitative subtraction instead of the use (sometimes artificially) of the complementation operator. Thus, a symbolic difference and a symbolic distance are defined to translate both the subtraction and the absolute value of the subtraction.

Formulas for Uncertainty Qualitative Theory

A qualitative multiplication of two degrees τ_α and τ_β is defined by the function MUL from $\mathcal{L}_M \times \mathcal{L}_M$ to $\mathcal{L}_M$ that verifies the properties of a t-norm to which are added the absorbent element τ_0 and the complementarity property: $\mathrm{MUL}(\tau_\alpha, \neg\tau_\alpha) = \tau_0$.

Similarly, a qualitative addition of two degrees τ_α and τ_β is a function ADD from $\mathcal{L}_M \times \mathcal{L}_M$ to $\mathcal{L}_M$ that verifies the properties of a t-conorm to which are added the absorbent element τ_{M-1} and the complementarity property: $\mathrm{ADD}(\tau_\alpha, \neg\tau_\alpha) = \tau_{M-1}$.

The qualitative subtraction of two degrees τ_α and τ_β such that $\tau_\beta \le \tau_\alpha$ is defined by the function SOUS from $\mathcal{L}_M \times \mathcal{L}_M$ to $\mathcal{L}_M$ that verifies the following properties: increasing relatively to the first argument; decreasing relatively to the second argument; SOUS has a neutral element τ_0 and the subtraction of two identical degrees gives the neutral element.

SOUS allows us to define an important axiom linking probability of the union to the probability of the intersection (see U6 below). SOUS corresponds in fact to the bounded difference of Zadeh, defined in the fuzzy logic framework.

The qualitative division of two degrees τ_α and τ_β and such that $\tau_\alpha \le \tau_\beta$ with $\tau_\beta \neq \tau_0$ is defined by the function DIV from $\mathcal{L}_M \times \mathcal{L}_M$ to $\mathcal{L}_M$ that verifies the following properties: increasing relatively to the first argument; decreasing relatively to the second argument; DIV has an absorbent element τ_0 and a neutral element τ_{M-1} and the division of two identical degrees, except for the absorbent element, gives the neutral element (boundary conditions).

Our choices for the four operators are the following:

1. $\mathrm{MUL}_{\mathcal{L}}(\tau_\alpha,\tau_\beta) = \neg(\tau_\alpha \rightarrow_L \neg\tau_\beta) = \tau_\gamma$ therefore $\gamma = \max(\alpha+\beta-(M-1),0)$

2. $\mathrm{ADD}_{\mathcal{L}}(\tau_\alpha,\tau_\beta) = (\neg\tau_\alpha \rightarrow_L \tau_\beta) = \tau_\delta$ therefore $\delta = \min(\alpha+\beta, M-1)$

3. $\mathrm{SOUS}_{\mathcal{L}}(\tau_\alpha,\tau_\beta) = \neg(\tau_\alpha \rightarrow_L \tau_\beta) = \tau_S$ therefore $S = \max(\alpha-\beta,0)$