Estimating Risk with Copulas

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INTRODUCTION

It is hardly possible to imagine the elaboration of any rational decision-making strategy in financial, capital, commodities markets, real sectors of the economy and international trade, without the proper consideration of the key concept of 'risk'. However, there is a lack of consensus in the literature with regard to a uniform and consistent definition of the concept 'risk', which successfully to capture all of its elements. For example, Machina & Rothschild (2008) discuss the fundamental difference between 'risk' and 'uncertainty' as follows: 'A situation is said to involve risk if the randomness facing an economic agent presents itself in the form of exogenously specified or scientifically calculable objective probabilities, as with gambles based on a roulette wheel or a pair of dice. A situation is said to involve uncertainty if the randomness presents itself in the form of alternative possible events, as with bets on a horse race, or decisions involving whether or not to buy earthquake insurance.' Similarly, according to ISO 31000:2009, a globally-accepted standard for risk management, authored by the International Standards Organization (2009), risk can be described as the 'effect of uncertainty on objectives'. In these two definitions, 'risk' is represented *as a symmetric* concept: with a potential for a gain or a loss. In contrast, McNeil, Frey, & Embrechts (2005) define an *asymmetric* version of the same concept as: 'the quantifiable likelihood of loss or less-than expected returns'. Borghesi & Gaudenzi (2013) formulate the concept in an analogous way as 'an unfavorable event capable of generating a

negative sign deviation from a given expected situation, such as a smaller gain or a greater loss' and as a 'set of hindrances that threaten the pursuit of the business's objectives'. These definitions with emphasis only on adverse results are adhered to in the regulatory risk management. Its central goal is to quantify the downside of risk, for example, in common tasks such as assessment of the decrease of the value of a portfolio, due to its exposure to various risk classes. A key risk that might adversely affect a portfolio is the market risk, defined as the overall uncertainty about the future asset price. For example, a press announcement for a merger between two key market players is supposed to affect their shares' prices. When the goal of the market participant is to assess his/her extra loss when a position must be quickly closed or changed due to transaction uncertainty, the situation involves assessment of liquidity risk. In contrast, credit risk involves inability of the second contractual party, for example a borrower or an issuer of corporate bonds, to meet his/her pre-established obligations. Counterparty risk is a subset of the credit risk category, as it covers cases where a second party of a specific transaction is unable to complete it at expiration. The emphasis of this chapter is exclusively on market risk, to which are exposed the returns of financial assets, and on copula-based estimation methods applied for one of the most popular market risk measures in quantitative risk management, Value at Risk (VaR). A second contribution is the presentation of the current experts' debate whether VaR can be successfully substituted by alternative risk measures.

BACKGROUND

Multivariate dependence among different asset classes and types is a topic in quantitative risk management that has attracted considerable attention in the scientific literature and is a particularly relevant topic in times of financial crises. There exist various methods for modelling dependence (see, e.g., Nelsen, 2006, for details). An obvious choice is the widely-used in financial applications Pearson's correlation coefficient ρ. It is a unit-free measure of the strength of *linear* relationship between two random variables X and Y in the interval $[-1,1]$. It is convenient, but not necessarily right to conclude that two risk factors are independent if $\rho = 0$. For example, let us suppose that the random variable X belongs to the symmetric around zero Student's t distribution, and let $Y = X^3$. By construction Y is completely dependent in a *non-linear* way on X, consequently ρ will be zero, and a wrong conclusion for independence between X and Y will be drawn. Among further discussed in Embrechts, McNeil, & Straumann (2002) and in McNeil et al. (2005) pitfalls of ρ is that it is defined only if the variances of X and Y are finite, or stated equivalently, when their distributions are not heavy-tailed. Consequently, not all values of ρ are attainable. In addition, ρ does not remain invariant under strictly increasing non-linear transformations of X and Y, because it contains information not only about the dependence between the participating two random variables, but also about their marginal behavior.

In recent years has evolved a more flexible statistical tool for tracking multivariate dependence. It is known as copula, derived from the Latin verb 'copulare', which means 'to connect'. Copulas are also referred to as 'bottom-up models' because they link the marginal distribution functions of a set of random variables and their multivariate distribution, and make it possible to calculate risk measures, such as VaR. For example, a recent proof of the superior performance of copulas compared to other bottom-up models and top-down models is the study of Ascheberg, Bick, & Kraft (2013).

Definition 1: (Copula properties, Joe, 1997; McNeil et al., 2005). A copula is a multivariate distribution function $C : [0,1]^d \to [0,1]$ with standard uniform marginal distributions $U_i \sim U(0,1)$ for $i = 1, 2, ..., d$ that satisfies the following properties:

$C(u_1, ..., u_d)$ is increasing in each u_i.

$C(1, ..., 1, u_i, 1, ..., 1) = u_i$ for $\forall i \in \{1, 2, ..., d\}$ where $u_i \in [0,1]$.

For $\forall (a_1, ..., a_d), \quad (b_1, ..., b_d) \in [0,1]^d$ with $a_i \le b_i$ is valid:

$$\sum_{i_1=1}^{2} ... \sum_{i_d=1}^{2} (-1)^{i_1 + ... + i_d} C(u_{1i_1}, ..., u_{di_d}) \ge 0$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$ for $\forall j \in \{1, 2, ..., d\}$.

1. Every copula has the following limits:

$$\max\left\{ \sum_{i=1}^{d} u_i + 1 - d, 0 \right\} \le C(u) \le \min\{u_1, ..., u_d\}.$$

Properties (1), (3) and (4) are satisfied for every multivariate distribution; (2) reflects the copula-specific property that the participating marginal distribution functions are standard uniform. However, the latter standardization, although universally adhered to in the financial literature, is not motivated by strict mathematical rules, but depends on the personal preferences. For example, in his 1940s research, Hoeffding used the interval $[-1/2, 1/2]$. The limits in (4) are known as the Fréchet–Hoeffding bounds (Figure 1). The right-hand side of (4) represents the comonotonicity copula (perfect positive dependence, or the upper bound), while the left-hand side is a copula (capturing perfect negative dependence, or the lower bound) only for dimension $d = 2$. On the basis of (4) can be drawn the conclusion that copulas as dependence structures can completely capture

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