Ranking Functions

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INTRODUCTION

Ranking functions have been introduced under the name of ordinal conditional functions in Spohn (1988; 1990). They are representations of epistemic states and their dynamics. The most comprehensive and up to date presentation is Spohn (manuscript).

BACKGROUND

The literature on knowledge, belief, and uncertainty in artificial intelligence is divided into two broad classes. In epistemic logic (Hintikka 1961, Halpern & Fagin & Moses & Vardi 1995), belief revision theory (Alchourrón & Gärdenfors & Makinson 1985, Gärdenfors 1988, Rott 2001), and nonmonotonic reasoning (Kraus & Lehmann & Magidor 1990, Makinson 2005) qualitative approaches are used to represent the epistemic state of an agent. In probability theory (Pearl 1988, Jeffrey 2004) and alternatives (Dempster 1968, Shafer 1976, Dubois & Prade 1988) epistemic states are represented quantitatively as degrees of belief rather than yes-or-no beliefs (see Halpern 2003 for an overview). One of the distinctive features of ranking functions is that they are quantitative, but nevertheless induce a notion of yes-or-no belief that satisfies the standard requirements of rationality, viz. consistency and deductive closure.

RANKING FUNCTIONS

Let W be a non-empty set of possibilities or worlds, and let **A** be a field of propositions over W. That is, **A** is a set of subsets of W that includes the empty set $\emptyset$ ($\emptyset \in \mathbf{A}$) and is closed under complementation with respect to W (if $A \in \mathbf{A}$, then $W \setminus A \in \mathbf{A}$) and finite intersection (if $A \in \mathbf{A}$ and $B \in \mathbf{A}$, then $A \cap B \in \mathbf{A}$). A function ρ from the field **A** over W into the natural numbers N extended by ∞, $\rho: \mathbf{A} \rightarrow N \cup \{\infty\}$, is a (*finitely minimitive*) *ranking function* on **A** if and only if for all propositions A, B in **A**:

1. $\rho(W) = 0$
2. $\rho(\emptyset) = \infty$
3. $\rho(A \cup B) = \min\{\rho(A), \rho(B)\}$

If the field of propositions **A** is closed under countable intersection (if $A_1 \in \mathbf{A}, \ldots, A_n \in \mathbf{A}, \ldots, n \in N$, then $A_1 \cap \ldots \cap A_n \cap \ldots \in \mathbf{A}$) so that **A** is a σ-field, a ranking function ρ on **A** is *countably minimitive* if and only if it holds for all propositions $A_1 \in \mathbf{A}, \ldots A_n \in \mathbf{A}, \ldots$

4. $\rho(A_1 \cup \ldots \cup A_n \cup \ldots) = \min\{\rho(A_1), \ldots, \rho(A_n), \ldots\}$

If the field of propositions **A** is closed under arbitrary intersection (if $\mathbf{B} \subseteq \mathbf{A}$, then $\cap \mathbf{B} \in \mathbf{A}$) so that **A** is a γ-field, a ranking function ρ on **A** is *completely minimitive* if and only if it holds for all sets of propositions $\mathbf{B} \subseteq \mathbf{A}$:

5. $\rho(\cup \mathbf{B}) = \min\{\rho(A): A \in \mathbf{B}\}$

A ranking function ρ on **A** is *regular* just in case $\rho(A) < \infty$ for each non-empty or consistent proposition A in **A**.

The conditional ranking function $\rho(\cdot|\cdot)$: $\mathbf{A} \times \mathbf{A} \rightarrow N \cup \{\infty\}$ based on the ranking function ρ on **A** is defined such that for all propositions A, B in **A**:

6. $\rho(A|B) = \rho(A \cap B) - \rho(B)$ if $A \neq \emptyset$, and $\rho(\emptyset|B) = \infty$

$\rho(\cdot|B)$ is a ranking function on **A**, for each proposition B in **A**.

A function κ from the set of worlds W into the natural numbers N, $\kappa: W \rightarrow N$, is a *pointwise ranking function* on W if and only if $\kappa(w) = 0$ for at least one world w in W. Each pointwise ranking function κ on W induces a regular and completely minimitive ranking function ρ_κ on every field of propositions **A** over W by defining

7. $\rho_\kappa(A) = \min\{\kappa(w): w \in A\}$ (= ∞ if $A = \emptyset$)

Huber (2006) discusses under which conditions a ranking function on a field of propositions **A** induces a pointwise ranking function on the underlying set of worlds W.

The rank of a proposition A, $\rho(A)$, represents the degree to which an agent with ranking function ρ disbelieves A. If $\rho(A) = 0$, the agent does not disbelieve A. However, this does not mean that she believes A. She may well suspend judgment and neither disbelieve A nor its complement or negation $W \setminus A$ (in this case $\rho(A) = \rho(W \setminus A) = 0$). Rather, belief in a proposition is characterized by disbelief in its negation: an agent with ranking function $\rho: \mathbf{A} \rightarrow N \cup \{\infty\}$ *believes* $A \in \mathbf{A}$ if and only if $\rho(W \setminus A) > 0$. The *belief set* Bel_ρ of an agent with ranking function $\rho: \mathbf{A} \rightarrow N \cup \{\infty\}$ is the set of all propositions she believes:

$$\text{Bel}_\rho = \{A \in \mathbf{A}: \rho(W \setminus A) > 0\}$$

The axioms of ranking theory require an agent to not disbelieve both a proposition and its negation – i.e. at least one of A, $W \setminus A$ has to be assigned rank 0. Thus an agent with ranking function $\rho: \mathbf{A} \rightarrow N \cup \{\infty\}$ believes $A \in \mathbf{A}$ if and only if $\rho(W \setminus A) > \rho(A)$. For a given $\rho: \mathbf{A} \rightarrow N \cup \{\infty\}$, this suggests to define the *belief function induced by* ρ, $\beta_\rho: \mathbf{A} \rightarrow Z \cup \{\pm\infty\}$, such that for all propositions A in **A**:

$$\beta_\rho(A) = \rho(W \setminus A) - \rho(A)$$

β_ρ assigns positive numbers to the propositions that are believed, negative numbers to the propositions that are disbelieved, and 0 to those propositions and their negations with respect to which the agent suspends judgment. As a consequence,

$$\text{Bel}_\rho = \{A \in \mathbf{A}: \beta_\rho(A) > 0\}$$

Bel_ρ is consistent and deductively closed in the finite sense, for every ranking function ρ on **A**. That is, $\cap\mathbf{B} \neq \varnothing$ for every finite $\mathbf{B} \subseteq \text{Bel}_\rho$; and $A \in \text{Bel}_\rho$ if there is a finite $\mathbf{B} \subseteq \text{Bel}_\rho$ such that $\cap\mathbf{B} \subseteq A$, for any $A \in \mathbf{A}$. If $\rho: \mathbf{A} \rightarrow N \cup \{\infty\}$ is countably/completely minimitive, Bel_ρ is consistent and deductively closed in the following countable/complete sense: $\cap\mathbf{B} \neq \varnothing$ for every countable/arbitrary $\mathbf{B} \subseteq \text{Bel}_\rho$; and $A \in \text{Bel}_\rho$ if there is a countable/arbitrary $\mathbf{B} \subseteq \text{Bel}_\rho$ such that $\cap\mathbf{B} \subseteq A$, for any $A \in \mathbf{A}$. As will be seen below, from a diachronic point of view the converse is true as well. However, first we have to discuss how an epistemic agent is to update her ranking function when she learns new information.

UPDATE RULES

A theory of epistemic states is incomplete if it does not account for the way the epistemic states are updated when the agent receives new information. As there are different formats in which the agent may receive new information, there are different update rules. The simplest and most unrealistic case is that of the agent becoming certain of a new proposition. This case is covered by

Plain Conditionalization

If the agent's epistemic state at time t is represented by the ranking function ρ *on* **A**, *and if, between t and t', the agent becomes certain of the proposition* $E \in \mathbf{A}$ *and of no logically stronger proposition* $E+ \subset E$, $E+ \in \mathbf{A}$, *then the agent's epistemic state at time t' should be represented by the ranking function* $\rho' = \rho(\cdot|E)$ *on* **A**.

We usually do not learn by becoming certain of a proposition, though. In most cases the new information merely changes the strength of our beliefs in various propositions. This is illustrated by a variation of an example due to Jeffrey (1983). Let our agent be interested in the color of the carpet of her hotel room. At time t, before checking in, she neither believes nor disbelieves any of the following three hypotheses: the carpet is beige (*beige*), the carpet is brown (*brown*), the carpet is black (*black*). However, she is certain that the carpet is either beige or brown or black. The relevant part of her ranking function at time t thus looks as follows: ρ(*beige*) = ρ(*not beige*) = ρ(*brown*) = ρ(*not brown*) = ρ(*black*) = ρ(*not black*) = ρ(*beige or brown or black*) = 0, ρ(*neither beige nor brown nor black*) = ∞.

At time t', after checking in and when opening the door to her room, it appears to the agent that the carpet is rather dark. As a consequence she now believes that the carpet is either brown or black. But since it is late at night, the curtains are closed, and she has not turned on the light yet, she cannot tell whether the carpet is brown or black. Her ranks for the relevant propositions

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