On Some New Fractional Type Heinz Inequalities

Yu Mei Liao, School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, China

Xuezhu Li, Department of Mathematics, Guizhou University, Guiyang, China

Jin Rong Wang, Department of Mathematics, Guizhou University, Guiyang, China

ABSTRACT

The authors shall discuss Heinz inequalities involving Riemann-Liouville fractional integrals for certain unitarily invariant norms. By using the convexity of function and fractional Hermit-Hadamard integral inequality, some refinements of Heinz inequalities are derived.

Keywords: Convexity of Function, Fractional Integrals, Heinz Inequalities, Riemann-Liouville, Unitarily Invariant Norm

1. INTRODUCTION

It is well known that Heinz means is one of special means which can interpolate between the geometric and the arithmetic mean. The Heinz means are defined as:

$$H_v(a, b) = \frac{a^v b^{-v} + a^{-v} b^v}{2}, \quad v \in [0, 1]$$  \hspace{1cm} (1)

for positive numbers a and b. For $v = 0.1$, the inequality (1) is equal to the arithmetic mean, and for $v = \frac{1}{2}$, that inequality (1) is equal to the geometric mean.

Moreover, one can show that:

$$\sqrt{ab} \leq H_v(a, b) \leq \frac{a + b}{2}$$

The general matrix version of the above asserts were proved in Bhatia and Davis (1993). Let A, B and X are three operators on a complex separable Hilbert space such that A and B are positive. For every unitarily invariant norm $\| \cdot \|$, the function:

$$f(v) = \| X^v A X^{1-v} + A^{1-v} X B^v \|, \quad v \in [0, 1]$$  \hspace{1cm} (2)

DOI: 10.4018/japuc.2013010102

Copyright © 2013, IGI Global. Copying or distributing in print or electronic forms without written permission of IGI Global is prohibited.
is convex on the interval \([0, 1]\), arrives its maximum at \(v = 0\) and \(v = 1\), attains its minimum at \(v = \frac{1}{2}\) and satisfies 
\[
f(v) = f(1 - v)
\]
for \(v \in [0, 1]\). Thus, we have the following standard operator version Heinz inequalities:

\[
2 \left\| \frac{1}{2} A^2 X B^2 \right\| \leq f(v) \leq \left\| AX + XB \right\| (3)
\]

It is easy to see that the old Heinz operator version inequality in Heinz (1951) is just the special case of the inequalities (3) whose norm is the operator bound norm.

For more refinements and applications on the inequalities (3), the reader can refer to Bhatia (2007), Hiai and Kosaki (1999), Hiai and Kosaki (2003), Kittaneh (2010), and Feng (2012) and references therein. Among these obtained results, the following known Hermit-Hadamard integral inequality for a convex function \(f\):

\[
(RL_a^\alpha + f)(x) = \frac{1}{\Gamma(a)} \int_a^x (x-t)^{a-1} f(t) dt, (0 \leq a < x \leq b)
\]

and:

\[
(RL_a^\alpha - f)(x) = \frac{1}{\Gamma(a)} \int_x^b (t-x)^{a-1} f(t) dt, (0 \leq a \leq x < b)
\]

respectively. Here \(\Gamma(\cdot)\) is the Gamma function.

Fractional calculus have recently proved to be a powerful tool for the study of dynamical properties of many interesting systems in physics, chemistry, and engineering. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more recent development on fractional calculus, one can see the monographs (Baleanu, Machado, & Luo, 2012; Diethelm, 2010; Kilbas, Srivastava, & Trujillo, 2006; Lakshmikantham, Leela, & Vasundhara...
Related Content

**Five Capabilities Model Applied to Multi-Robot Systems**
[www.irma-international.org/article/five-capabilities-model-applied-to-multi-robot-systems/131459/](www.irma-international.org/article/five-capabilities-model-applied-to-multi-robot-systems/131459/)

**A Novel Detection Method of Paper Defects Based on Visual Attention Mechanism**
Ping Jiang and Tao Gao (2013). *Global Applications of Pervasive and Ubiquitous Computing* (pp. 159-165).
[www.irma-international.org/chapter/novel-detection-method-paper-defects/72940/](www.irma-international.org/chapter/novel-detection-method-paper-defects/72940/)

**Spontaneous Service-Providing using WS4D in Smart Environments**
[www.irma-international.org/article/spontaneous-service-providing-using-ws4d-in-smart-environments/130644/](www.irma-international.org/article/spontaneous-service-providing-using-ws4d-in-smart-environments/130644/)

**An Exploration about Krashen's Input Hypothesis in the Computer Network Environment**

**Mobile Devices: Designing Hybrid Body-Spaces**
[www.irma-international.org/article/mobile-devices-designing-hybrid-body/43585/](www.irma-international.org/article/mobile-devices-designing-hybrid-body/43585/)