A Family of Superstable n-D Mappings

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ABSTRACT

In this article it is rigorously shown that a family of n-D mappings is superstable for some ranges in its bifurcation parameters space. This result is also tested numerically through a 3-D mapping, which has a simple form and complicated behaviors.

Keywords: Chaos, Co-Existence Phenomena, Dynamical System, Lyapunouv Exponent, Superstability

INTRODUCTION

The superstability of a dynamical motion is defined with existence of a minus infinity Lyapunouv exponent, this mean that this motion is attractive. There are several methods for constructing 1-D polynomial mappings with attracting cycles or superstable cycles (Zhang & Agarwal, 2003; Liu, Zhang, & Liu, in press) based on Lagrange and Newton interpolations. Superstable phenomena in some 1-D maps embedded in circuits and systems are studied in Matsuoka and Saito (2007) and Saito, Matsuoka, Ishikawa, and Ishige (2007) these maps are obtained from the study of nonautonomous piecewise constant circuits and biological models (Matsuoka & Saito, 2006a, 2006b; Matsuoka, Torikai, & Saito, 2006; Matsuoka, Saito, & Torikai, 2005, 2006; Keener & Glass, 1984). Rich dynamical behaviors can be seen in the presence of superstability (Matsuoka & Saito, 2006a; Matsuoka, Saito, et al., 2005, 2006), especially, the attractivity of the motion that guaranty its stability. The property of superstability is probably rare in n-D dynamical systems with \( n \geq 2 \). Also, superstability is a local property in the space of bifurcation parameters, i.e., in general, not all the behaviors of the considered system are superstable. For \( n=1 \), this property is well invistigated analytically and numerically for discrete time systems (Zhang & Agarwal, 2003; Liu, Zhang, & Liu, in press; Matsuoka & Saito, 2007; Saito et al., 2007).

The essential motivation of the present work is the construction of a family of superstable n-D mappings in the sense that all the states of a member of this family are superstable, i.e. they have a minus infinity Lyapunouv exponent for some regions in the bifurcation parameters space of these mappings. Indeed, let us consider the following n-D map given by:
\[
\mathbf{f}(X, \rho) = \begin{pmatrix}
    f_1(x_1, x_2, \ldots, x_{n-1}, \rho) \\
    f_2(x_1, x_2, \ldots, x_{n-1}, \rho) \\
    \vdots \\
    f_n(x_1, x_2, \ldots, x_{n-1}, x_n, \rho)
\end{pmatrix}
\]  

(1)

where \( X = (x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n \) and \( \rho = (\rho_1, \rho_2, \ldots, \rho_k) \in \mathbb{R}^k \) is the bifurcation parameter. Assuming that the vector field \( \mathbf{f} \) of the map (1) is continuously differentiable in a region of interest, and it has at least one fixed point, so bounded states are possible, i.e., the system of algebraic equations given by

\[
\begin{pmatrix}
    f_1(x_1, x_2, \ldots, x_{n-1}, \rho) = x_1 \\
    f_2(x_1, x_2, \ldots, x_{n-1}, \rho) = x_2 \\
    \vdots \\
    f_n(x_1, x_2, \ldots, x_{n-1}, x_n, \rho) = x_3
\end{pmatrix}
\]  

(2)

has at least one real solution.

The goal of this article is the rigorous proof that the n-D maps given by equation (1) have minus infinity Lyapunov exponent for some \( \rho \in \mathbb{R}^k \) under some realizable conditions. A numerical example is also given and discussed. We remark that the first \( n-1 \) components of the vector field \( \mathbf{f} \) of the map (1) does not depend on the last variable \( x_n \), and the last component \( f_n \) must be depend on the variable \( x_n \) in which the map (1) is not trivial.

The Jacobian matrix of the map (1) is given by

\[
D\mathbf{f}(X, \rho) = \begin{pmatrix}
    f_{11} & f_{12} & \ldots & f_{1n-1} & 0 \\
    f_{21} & f_{22} & \ldots & f_{2n-1} & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    f_{(n-1)1} & f_{(n-1)2} & \ldots & f_{(n-1)(n-1)} & 0 \\
    f_{n1} & f_{n2} & \ldots & f_{n(n-1)} & f_{nn}
\end{pmatrix}
\]  

(3)

\[
f_{ij} = \frac{\partial f_i(X, \rho)}{\partial x_j} = f_i(X, \rho),
\]

\( i, j = 1, 2, \ldots, n \), and \( f_{nn} = f_n(X, \rho) \), where \( X = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \). On the other hand, let us consider the following n-D dynamical system:

\[
X_{i+1} = g(X_i), X_i \in \mathbb{R}^n, i = 0, 1, 2, \ldots
\]  

(4)

where the function \( g: \mathbb{R}^n \to \mathbb{R}^n \) is the vector field associated with system (3) and \( X_i = (x_i^1, x_i^2, \ldots, x_i^n) \in \mathbb{R}^n \) defining the iterations of the system. Let \( J(X_i) \) be its Jacobian evaluated at \( X_i \in \mathbb{R}^n, i = 0, 1, 2, \ldots \), and define the matrix:

\[
T_N(X_0) = J(X_{N-1})J(X_{N-2})\ldots J(X_1)J(X_0)
\]  

(5)

Moreover, let \( J_i(X_0, N) \) be the modulus of the \( i \)-th eigenvalue of the \( N \)-th matrix. \( T_N(X_0) \), where \( i = 1, 2, \ldots, n \) and \( N = 0, 1, 2, \ldots \).

Now, the Lyapunov exponents for a n-D discrete-time system are defined by:

\[
l_N(X_0) = \ln \left( \lim_{N \to \infty} |J_i(X_0, N)|^{\frac{1}{N}}, i = 1, 2, \ldots, n \right)
\]  

(6)

Based on this definition, we will give a rigorous proof that the family of n-D maps given by equation (1) is superstable for all its bifurcation parameters and all initial conditions. We have

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