One vs.Two vs. Multidimensional Searches for Optimization Methods

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INTRODUCTION

Optimization is one of the most widely used tools in operations research and have been used by decision makers of public and private organizations to make smarter decisions. Optimization is used daily to solve numerous real-world complex problems. For instance, the most energy efficient way in scheduling crude oil operations (Wu et al., 2017), the design of decision-support tools for outpatient appointment systems (Ahmadi-Javid et al., 2017), the optimal routes for drone delivery (Dorling at al., 2017), cancer treatment options (Biesecker et al., 2010), and vaccination campaigns (Matrajt et al., 2021) can all be determined by optimization models.

Some important classes of optimization models include linear, nonlinear, and integer programming. This chapter focuses primarily on linear programming, but some insights for nonlinear optimization is discussed. Formally, define a linear program (LP) as:

maximize $z=c^Tx$
subject to: $Ax\leq b$
$x\geq 0$

where $n,m\in\mathbb{Z}_+$, $x\in\mathbb{R}^n_+$, $c\in\mathbb{R}^n$, $A\in\mathbb{R}^{m\times n}$, and $b\in\mathbb{R}^m$. Let the polyhedron $S=\left\{x\in\mathbb{R}^n_+ : Ax\leq b\right\}$ be the set of feasible solutions of an LP and (x^*,z^*) be its optimal solution where $x^*\in S$ and $z^*=c^Tx^*\geq c^Tx'$ for all $x'\in S$.

Finding an optimal solution to LPs in a reasonable amount of time is vital for decision makers. Even though algorithms to solve LPs in polynomial time exist (Gondzio, 2012), a substantial amount of time can still be required to solve them. This is because real-world applications are often large, sparse, and dependent on a substantial amount of data. In fact, the amount of data to model optimization problems has drastically increased over the recent years due to the discovery of many efficient data analytics techniques (Vitor, 2019). Therefore, finding methods to solve LPs more quickly can benefit numerous organizations in obtaining better solutions to their problems.

Most of linear programming (and nonlinear programming) algorithms are considered one-dimensional search. That is, from a solution, a single search direction is selected and a one-dimensional subproblem is solved to determine how far to move along this direction. A new solution is obtained and the process repeats. Figure 1 demonstrates this concept using a simplex framework. For brevity, one can view a simplex framework as an algorithm that moves between solutions that are located at the vertices of the polyhedron. On the other hand, an interior point framework moves between solutions within the interior of the polyhedron. For each iteration k, an improving search direction d^k is selected from x^k.

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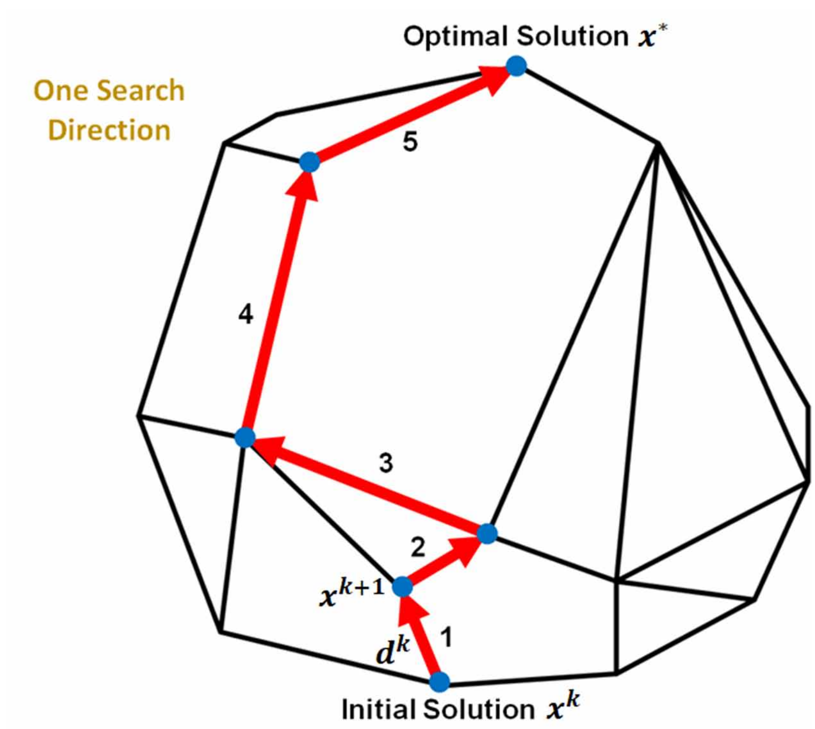
Optimally solving a one-dimensional subspace LP defined by $x^k+\lambda^k d^k$ and $\lambda^k>0$ determines how far to move from x^k along d^k. The optimal solution to this subproblem is λ^{k^*} and x^{k+1} is computed as $x^{k+1}=x^k+\alpha\lambda^{k^*}d^k$ where $0<a<1$. Notice that solving a one-dimensional subspace problem is the same as performing a one-dimensional search.

Figure 1. Graphical representation of one-dimensional search algorithms



Multidimensional search methods have also been developed. Differently than the traditional one-dimensional search techniques, these methods consider more than one search direction at each step. Furthermore, a multidimensional subproblem must be solved to determine how far to move along each direction. Figure 2 graphically presents this concept. For simplicity, two search directions are considered. From x^k, two search directions, $\left(d_1^k, d_2^k\right)$, are selected at each step k (at least one must be improving). Both search directions define a plane, and this plane when intersected with the polyhedron creates a two-dimensional subspace. Optimally solving a two-dimensional subspace LP defined by $x^k+\lambda_1^k d_1^k+\lambda_2^k d_2^k$, $\lambda_1^k>0$, and $\lambda_2^k>0$ determines the step length. The optimal solution to this sub-problem, let say $\left(\lambda_1^{k^*}, \lambda_2^{k^*}\right)$, determines the next solution $x^{k+1}=x^k+\alpha\left(\lambda_1^{k^*}d_1^k+\lambda_2^{k^*}d_2^k\right)$ where $0<a<1$. Again, solving a two-dimensional subspace problem is the same as performing a two-dimensional search (or a multidimensional search). If more than two search directions are considered, let say d_i^k for all $i\in\{1,2\ldots t\}$ where t is some integer upper bound, then the multidimensional subspace LP becomes $x^k+\sum_{i=1}^{t}\lambda_i^k d_i^k$ and $\lambda_i^k>0$ for all $i\in\{1,2\ldots t\}$ and the next solution is $x^{k+1}=x^k+\alpha\sum_{i=1}^{t}\lambda_i^k d_i^k$ where $0<a<1$.