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On Concept Algebra:
A Denotational Mathematical Structure for Knowledge and Software Modeling

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ABSTRACT

Concepts are the most fundamental unit of cognition that carries certain meanings in expression, thinking, reasoning, and system modeling. In denotational mathematics, a concept is formally modeled as an abstract and dynamic mathematical structure that encapsulates attributes, objects, and relations. The most important property of an abstract concept is its adaptive capability to autonomously interrelate itself to other concepts. This article presents a formal theory for abstract concepts and knowledge manipulation known as "concept algebra." The mathematical models of concepts and knowledge are developed based on the object-attribute-relation (OAR) theory. The formal methodology for manipulating knowledge as a concept network is described. Case studies demonstrate that concept algebra provides a generic and formal knowledge manipulation means, which is capable to deal with complex knowledge and software structures as well as their algebraic operations.

Keywords: *case studies; cognitive informatics; concept; concept algebra; denotational mathematics; knowledge; knowledge network; learning; mathematics; natural intelligence; the OAR model*

INTRODUCTION

In cognitive informatics, logic, linguistics, psychology, software engineering, and knowledge engineering, concepts are identified as the basic unit of both knowledge and reasoning (Anderson, 1983; Colins & Loftus, 1975; Ganter & Wille, 1999; Hampton, 1997; Hurley, 1997; Matlin, 1998; Murphy, 1993; Wang, 2006a, 2006b, 2006c, 2007a, 2007c; Wang & Wang, 2006; Wilson & Keil, 1999). The rigorous modeling and formal treatment of concepts are at the center of theories for knowledge presentation and manipulation (Smith & Medin, 1981; Wille, 1982; Murphy, 1993; Codin, Missaoui, & Alaoui, 1995; Wilson & Keil, 1999; Yao, 2004; Chen & Yao, 2005). A *concept* in linguistics is a noun or noun-phrase that serves as the subject of a *to-be* statement (Hurley, 1997; Wang, 2002a, 2006a, 2006c, 2007d). Concepts in cognitive informatics (Wang, 2002a, 2006c, 2007b, 2007e) are an abstract structure that carries certain meaning in almost all cognitive processes such as thinking, learning, and reasoning.

Definition 1. *A concept is a cognitive unit to identify and/or model a real-world concrete entity and a perceived-world abstract subject.*

Based on concepts and their relations, meanings of real-world concrete entities may be represented and semantics of abstract subjects may be embodied. Concepts can be classified into two categories, known as the *concrete* and *abstract* concepts. The former are proper concepts that identify and model real-world entities such as the sun, a pen, and a computer. The latter are virtual concepts that identify and model abstract subjects, which cannot be directly mapped to a real-world entity, such as the mind, a set, and an idea. The abstract concepts may be further classified into *collective* concepts, such as collective nouns and complex concepts, or *attributive* concepts such as qualitative and quantitative adjectives. The concrete concepts are used to embody meanings of subjects in reasoning while the abstract concepts are used as intermediate representatives or modifiers in reasoning.

A concept can be identified by its intension and extension (Hurley, 1997; Smith & Medin, 1981; Wang, 2006c; Wille, 1982; Yao, 2004).

Definition 2. *The intension of a concept is the attributes or properties that a concept connotes.*

Definition 3. *The extension of a concept is the members or instances that the concept denotes.*

For example, the intension of the concept *pen* connotes the attributes of being a writing tool, with a nib, and with ink. The extension of the pen denotes all kinds of pens that share the common attributes as specified in the intension of the concept, such as a ballpoint pen, a fountain pen, and a quill pen.

In computing, a concept is an identifier or a name of a class. The intension of the class is a set of operational attributes of the class. The extension of the class is all its instantiations or objects and derived classes. Concept algebra provides a rigorous mathematical model and a formal semantics for object-oriented class modeling and analyses. The formal modeling of computational classes as a dynamic concept with predesigned behaviors may be referred to "system algebra" (Wang, 2006b, 2007d, 2008b, 2008d).

This article presents a formal treatment of abstract concepts and an entire set of algebraic operations on them. The mathematical model of concepts is established first. Then, the abstract mathematical structure, *concept algebra*, is developed for knowledge representation and manipulation. Based on concept algebra, a knowledge system is formally modeled as a concept network, where the methodology for knowledge manipulating is presented. Case studies demonstrate that concept algebra provides a denotational mathematical means for manipulating complicated abstract and concrete knowledge structures as well as their algebraic operations.

THE MATHEMATICAL MODEL OF ABSTRACT CONCEPTS

This section describes the formal treatment of abstract concepts and a new mathematical structure known as concept algebra in cognitive informatics and knowledge engineering. Before an abstract concept is defined, the semantic environment or context (Chen & Yao, 2005; Ganter & Wille, 1999; Hampton, 1997; Hurley, 1997; Medin & Shoben, 1988) in a given language, is introduced.

Definition 4. *Let $\mathcal{O}$ denote a finite or infinite nonempty set of objects, and $\mathcal{A}$ be a finite or infinite nonempty set of attributes, then a semantic environment or context Θ is denoted as a triple, i.e.:*

$$\begin{aligned}\Theta &\triangleq (\mathcal{O}, \mathcal{A}, \mathcal{R}) \\ &= \mathcal{R}: \mathcal{O} \to \mathcal{O} | \mathcal{O} \to \mathcal{A} | \mathcal{A} \to \mathcal{O} | \mathcal{A} \to \mathcal{A}\end{aligned} \tag{1}$$

where $\mathcal{R}$ is a set of relations between $\mathcal{O}$ and $\mathcal{A}$, and | demotes alternative relations.