Isac's Cones in General Vector Spaces

Vasile Postolică
Romanian Academy of Scientists,"Vasile Alecsandri" University of Bacău, Faculty of Sciences, Bacău, România

INTRODUCTION

As we have already seen in (Bahya, A. O., 1989), (Hyers, D. H., Isac, G., & Rassias, T. M., 1997), (Isac, G., 1981, 1983, 1985, 1994, 1998, 2003), (Isac, G. et al., 1993, 2002, 2003, 2005), a kind of orders in locally convex spaces is represented by the ordering cones induced by Isac's cones and defined following the domination of the seminorms by linear continuous functionals on them. Essentially, the property involved in the definition of Isac's cones is metrical, not topological. The content of this research work is organized as follows: Section 2 is dedicated to Isac's cones in locally convex spaces. The extension in real or complex vector spaces is presented in Section 3, completed by pertinent examples and significant comments. Section 4 is devoted to the efficiency, optimization and to our important coincidence result between the efficient points sets and Choquet's boundaries. In Section 5, we formulate some open problems. Finally, we indicate future research directions, concluding remarks, acknowledgements and the selected bibliography which refers only to the papers which were used to complete this paper.

ISAC'S CONES IN LOCALLY CONVEX SPACES

First of all, we remember some usual notions and results concerning the linear (topological) spaces. A linear topology on a real or complex vector space X means any topology τ on X with respect to which the addition and the scalar multiplication are continuous, the couple (X, τ) being named a topological vector space. If one denotes $\Gamma = R$ or $\Gamma = C$, then any function $p : X \to R_+$ satisfying the following properties:

$$p(\chi x) = \left|\chi\right| p(x)$$

for all $x \in X$, $\chi \in \Gamma$ (the absolute homogeneity) and

$$p(x + y) \leq p(x) + p(y)$$

(the triangle inequality) is called seminorm on X. Every linear topology is a locally convex topology iff it is generated by a family of seminorms as follows: let

$$P = \left\{p_\alpha : \alpha \in I\right\}$$

be a family of seminorms defined on X. For every $x \in X, \varepsilon > 0$ and $n \in N^*$ let

$$V(x; p_1, p_2, \ldots, p_n; \varepsilon) = \{y \in X : p_\alpha(y - x) < \varepsilon, \forall \alpha = \overline{1, n}\}$$

Then, the family

$$\varsigma_0(x) = \left\{ \begin{matrix} V(x; p_1, p_2, \ldots, p_n; \varepsilon) : \\ n \in N^*, p_\alpha \in P, \alpha = \\ \overline{1, n}, \varepsilon > 0\} \end{matrix} \right\}$$

has the next properties:

$$\left(V_1\right) x \in V, \forall V \in V_0(x);$$

DOI: 10.4018/978-1-4666-5202-6.ch121

$\left(V_2\right) \forall V_1, V_2 \in V_0(x),$

$\exists V_3 \in V_0(x);$

$V_3 \subseteq V_1 \cap V_2;$

$\left(V_3\right) \forall V \in V_0(x),$

$\exists U \in V_0(x),$

$U \subseteq V$ such that $\forall y \in U, \exists W \in V_0(y)$ with $W \subseteq V$.

Therefore, $V_0(x)$ is a base of neighborhoods for x and taking

$\varsigma(x) = \{V \subseteq X : \exists U \in V_0(x),$

$U \subseteq V\}$, the set

$\tau = \{D \subseteq X : D \in \varsigma, (x)$

$\forall x \in D\} \cup \left\{\varnothing\right\}$

is the locally convex topology generated by the family P and the pair (X, τ) is called a locally convex space. Obviously, the usual operations which induce the structure of linear space on X are continuous with respect to this topology. The corresponding topological space (X, τ) is a Hausdorff locally convex space iff the family P is sufficient, that is,

$\forall\ x_0 \in X \setminus \{\theta\}, \exists p_\alpha \in P$

with $p_\alpha(x_0) \neq 0$. In this research paper we will suppose that the space (X, τ) sometimes denoted by X is a Hausdorff locally convex space. Every non-empty subset K of X satisfying the following properties: $K + K \subseteq K$ and

$\chi K \subseteq K, \forall \chi \in R_+$

is named convex cone (wedge). If, in addition, $K \cap K = \left\{\theta\right\}$, then K is called pointed or with vertex at zero. Whenever K satisfies the last three conditions and $x \in X$, then the set $x + K$ is called cone with vertex x. Clearly, any (pointed) convex cone K in X generates an (ordering) pre-ordering on X defined by

$x \leq y (x, y \in X)$

if $y - x \in K$. If X^* is the topological dual of X, that is, the collection of all linear continuous functionals on X, then the dual cone of K is defined by

$$K^* = \left\{x^* \in X^* : x^*(x) \geq 0, \forall x \in K\right\}$$

and its corresponding polar is $K^0 = -K^*$. The weak topology on X identified as the smallest locally convex topology for which every $x^* \in X^*$ is continuous is generated by the family

$$\left\{p_{x^*} : x^* \in X^*\right\}$$

where

$$p_{x^*}(x) = \left|x^*(x)\right|,$$

$$\forall x^* \in X^*, x \in X.$$

We recall that a pointed convex cone $K \subset X$ is normal with respect to the topology defined by P if it fulfils one of the next equivalent assertions:

1. There exists at a base Ω of neighborhoods for the origin θ in X such that

$$V = (V + K) \cap (V - K), \forall V \in \Omega;$$