Neural/Fuzzy Computing Based on Lattice Theory

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INTRODUCTION

Computational Intelligence (CI) consists of an evolving collection of methodologies often inspired from nature (Bonissone, Chen, Goebel & Khedkar, 1999, Fogel, 1999, Pedrycz, 1998). Two popular methodologies of CI include neural networks and fuzzy systems.

Lately, a unification was proposed in CI, at a "data level", based on lattice theory (Kaburlasos, 2006). More specifically, it was shown that several types of data including vectors of (fuzzy) numbers, (fuzzy) sets, 1D/2D (real) functions, graphs/trees, (strings of) symbols, etc. are partially(lattice)-ordered. In conclusion, a unified cross-fertilization was proposed for knowledge representation and modeling based on lattice theory with emphasis on clustering, classification, and regression applications (Kaburlasos, 2006).

Of particular interest in practice is the totally-ordered lattice $(\mathsf{R},\leq)$ of real numbers, which has emerged historically from the conventional measurement process of successive comparisons. It is known that $(\mathsf{R},\leq)$ gives rise to a hierarchy of lattices including the lattice $(\mathsf{F},\leq)$ of *fuzzy interval numbers*, or FINs for short (Papadakis & Kaburlasos, 2007).

This article shows extensions of two popular neural networks, i.e. *fuzzy-ARTMAP* (Carpenter, Grossberg, Markuzon, Reynolds & Rosen 1992) and *self-organizing map* (Kohonen, 1995), as well as an extension of conventional *fuzzy inference systems* (Mamdani & Assilian, 1975), based on FINs. Advantages of the aforementioned extensions include both a capacity to rigorously deal with nonnumeric input data and a capacity to introduce tunable nonlinearities. *Rule induction* is yet another advantage.

BACKGROUND

Lattice theory has been compiled by Birkhoff (Birkhoff, 1967). This section summarizes selected results regarding a Cartesian product lattice $(\mathsf{L},\leq)=(\mathsf{L}_1,\leq_1)\times\ldots\times(\mathsf{L}_N,\leq_N)$ of *constituent* lattices $(\mathsf{L}_i,\leq_i)$, i=1,…,N.

Given an *isomorphic* function θ_i: $(\mathsf{L}_i,\leq_i)\to(\mathsf{L}_i,\leq_i)^\partial$ in a constituent lattice $(\mathsf{L}_i,\leq_i)$, i=1,…,N, where $(\mathsf{L}_i,\leq_i)^\partial \equiv (\mathsf{L}_i,\leq_i^\partial)$ denotes the *dual* (lattice) of lattice $(\mathsf{L}_i,\leq_i)$, then an isomorphic function θ: $(\mathsf{L},\leq)\to(\mathsf{L},\leq)^\partial$ is given by $\theta(x_1,\ldots,x_N)=(\theta_1(x_1),\ldots,\theta_N(x_N))$.

Given a *positive valuation* function v_i: $(\mathsf{L}_i,\leq_i)\to\mathsf{R}$ in a constituent lattice $(\mathsf{L}_i,\leq_i)$, i=1,…,N then a positive valuation v: $(\mathsf{L},\leq)\to\mathsf{R}$ is given by $v(x_1,\ldots,x_N)=v_1(x_1)+\ldots+v_N(x_N)$.

It is well-known that a positive valuation v_i: $(\mathsf{L}_i,\leq_i)\to\mathsf{R}$ in a lattice $(\mathsf{L}_i,\leq_i)$ implies a metric function d_i: $\mathsf{L}_i\times\mathsf{L}_i\to\mathsf{R}_0^+$ given by $d_i(a,b) = v_i(a\vee b) - v_i(a\wedge b)$.

Minkowski metrics d_p: $(\mathsf{L}_1,\leq_1)\times\ldots\times(\mathsf{L}_N,\leq_N)=(\mathsf{L},\leq)\to\mathsf{R}$ are given by

$$d_p(x,y) = \left[d_1^p(x_1,y_1)+\ldots+d_N^p(x_N,y_N)\right]^{1/p},$$

where

$x= (x_1,\ldots,x_N)$, $y=(y_1,\ldots,y_N)$, $p\in\mathsf{R}$.

An *interval* $[a,b]$ in a lattice $(\mathsf{L},\leq)$ is defined as the set $[a,b]\doteq\{x\in\mathsf{L}: a\leq x\leq b, a,b\in\mathsf{L}\}$. Let $\tau(\mathsf{L})$ denote the set of intervals in a lattice $(\mathsf{L},\leq)$. It turns out that $(\tau(\mathsf{L}),\leq)$ is a lattice, ordered by set inclusion.

Definition 1. The *size* Z_p: $\tau(\mathsf{L})\to\mathsf{R}_0^+$ of a lattice $(\mathsf{L},\leq)$ interval $[a,b]\in\tau(\mathsf{L})$, with respect to a positive valuation v: $(\mathsf{L},\leq)\to\mathsf{R}$, is defined as $Z_p([a,b])=d_p(a,b)$.

NEURAL/FUZZY COMPUTING BASED ON LATTICE THEORY

This section delineates modified extensions to a hierarchy of lattices stemming from the totally ordered lattice $(\mathsf{R},\leq)$ of real numbers. Then, it details the relevance of

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novel mathematical tools. Next, based on the previous mathematical tools, this section presents extensions of ART/SOM/FIS. Finally, it discusses comparative advantages.

Modified Extensions in a Hierarchy of Lattices

Consider the product lattice $(\Delta,\leq) = (\mathsf{R}\times\mathsf{R},\leq^{\partial}\times\leq) = (\mathsf{R}\times\mathsf{R},\geq\times\leq)$ of *generalized intervals*. A *generalized interval* (element in Δ) will be denoted by $[a,b]$ and will be called *positive* (*negative*) for $a\leq b$ ($a>b$). The set of positive (negative) generalized intervals will be denoted by Δ_+ (Δ_-) – We remark that the set of *positive* generalized intervals is isomorphic to the set of *conventional intervals* in the set R of real numbers.

A *decreasing* function θ_{R}: $\mathsf{R}\to\mathsf{R}$ is an isomorphic function θ_{R}: $(\mathsf{R},\leq)\to(\mathsf{R},\leq)^{\partial}$; furthermore, a *strictly increasing* function v_{R}: $\mathsf{R}\to\mathsf{R}$ is a positive valuation v_{R}: $(\mathsf{R},\leq)\to\mathsf{R}$. Hence, function v_{Δ}: $(\Delta,\leq)\to\mathsf{R}$ given by $v_{\Delta}([a,b])= v_{\mathsf{R}}(\theta_{\mathsf{R}}(a))+v_{\mathsf{R}}(b)$ is a positive valuation in lattice $(\Delta,\leq)$. There follows a metric function d_{Δ}: $\Delta\times\Delta\to\mathsf{R}_0^+$ given by $d_{\Delta}([a,b],[c,d])=[v_{\mathsf{R}}(\theta_{\mathsf{R}}(a\wedge c))-v_{\mathsf{R}}(\theta_{\mathsf{R}}(a\vee c))] + [v_{\mathsf{R}}(b\vee d)-v_{\mathsf{R}}(b\wedge d)]$; in particular, for $\theta_{\mathsf{R}}(x)= -x$ and $v_{\mathsf{R}}(x)= x$ it follows $v_{\Delta}([a,b])= |a-c| + |b-d|$. Choosing *parametric* functions $\theta_{\mathsf{R}}(.)$ and $v_{\mathsf{R}}(.)$ there follow tunable nonlinearities in lattice $(\mathsf{R},\leq)$. Moreover, note that Δ is a *real linear space* with

- *addition* defined as $[a,b] + [c,d] = [a+c,b+d]$, and
- *multiplication* (by a real k) defined as $k[a,b] = [ka,kb]$.

It turns out that Δ_+ (as well as Δ_-) is *cone* in linear space Δ – Recall that a subset C of a linear space is called *cone* if for all $x\in C$ and $\lambda>0$, we have $\lambda x\in C$.

Definition 2. A *generalized interval number* (*GIN*) is a function f: $(0,1]\to\Delta$.

Let G denote the set of GINs. It follows that $(\mathsf{G},\leq)$ is a lattice, in particular $(\mathsf{G},\leq)$ is the Cartesian product of lattices $(\Delta,\leq)$. Moreover, G is a real linear space with

- *addition* defined as $(G_1 + G_2)(h) = G_1(h) + G_2(h)$, $h\in(0,1]$, and
- *multiplication* (by a real k) defined as $(kG)(h) = kG(h)$, $h\in(0,1]$.

We remark that the cardinality of set G equals $\aleph_1^{\aleph_1} = \left(2^{\aleph_0}\right)^{\aleph_1} = 2^{\aleph_0\aleph_1} = 2^{\aleph_1} = \aleph_2 > \aleph_1$, where $\aleph_1$ is the cardinality of the set R of real numbers.

Proposition 3. Consider metric(s) d_{Δ}: $\Delta\times\Delta\to\mathsf{R}_0^+$ in lattice $(\Delta,\leq)$. Let $G_1,G_2\in(\mathsf{G},\leq)$. Assuming that the following integral exists, a metric function d_{G}: $\mathsf{G}\times\mathsf{G}\to\mathsf{R}_0^+$ is given by

$$d_{\mathsf{G}}(G_1,G_2) = \int_0^1 d_{\Delta}(G_1(h),G_2(h))dh.$$

Our interest here focuses on the *sublattice* $(\mathsf{F},\leq)$ of lattice $(\mathsf{G},\leq)$, namely sublattice of *fuzzy interval numbers* (*FIN*s). A FIN is defined rigorously as follows.

Definition 4. A *fuzzy interval number* (*FIN*) F is a GIN such that either (1) both $F(h)\in\Delta_+$ and $h_1\leq h_2 \Rightarrow F(h_1)\geq F(h_2)$, for all $h\in(0,1]$ (*positive FIN*) or (2) there is a positive FIN P such that $F(h) = -P(h)$, for all $h\in(0,1]$ (*negative FIN*).

Let F_+ (F_-) denote the set of positive (negative) FINs. Note that both $\mathsf{F}_+\cup\mathsf{F}_- = \mathsf{F}$ and $\mathsf{F}_+\cap\mathsf{F}_-=\varnothing$ hold. Furthermore, F_+ (F_-) is a cone with cardinality $\aleph_1$ (Kaburlasos & Kehagias, 2006). The previous mathematical analysis may potentially produce useful techniques based on lattice vector theory (Vulikh, 1967). A *positive FIN* will simply be called "FIN". A FIN may admit different interpretations including a (fuzzy) number, an interval, and a cumulative distribution function.

Relevance of Novel Mathematical Tools

A fundamental mathematical result in *fuzzy set theory* is the "resolution identity theorem", which states that a fuzzy set can, equivalently, be represented either by its membership function or by its α-cuts (Zadeh, 1975). The aforementioned theorem has been given little attention in practice to date. However, some authors have capitalized on it by designing effective as well as efficient fuzzy inference systems (FIS) involving fuzzy numbers whose α-cuts are conventional closed intervals (Uehara & Fujise, 1993, Uehara & Hirota, 1998).

This work builds on the abovementioned mathematical result as follows. In the first place, we drop the possibilistic interpretation of a membership function. Then, we consider the corresponding "α-cuts representation".

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