# Fuzzy Graphs and Fuzzy Hypergraphs

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## INTRODUCTION

Graph theory has numerous application to problems in systems analysis, operations research, economics, and transportation. However, in many cases, some aspects of a graph-theoretic problem may be uncertain. For example, the vehicle travel time or vehicle capacity on a road network may not be known exactly. In such cases, it is natural to deal with the uncertainty using the methods of fuzzy sets and fuzzy logic.

Hypergraphs (Berge,1989) are the generalization of graphs in case of set of multiarity relations. It means the expansion of graph models for the modeling complex systems. In case of modelling systems with fuzzy binary and multiarity relations between objects, transition to fuzzy hypergraphs, which combine advantages both fuzzy and graph models, is more natural. It allows to realise formal optimisation and logical procedures.

However, using of the fuzzy graphs and hypergraphs as the models of various systems (social, economic systems, communication networks and others) leads to difficulties. The graph isomorphic transformations are reduced to redefinition of vertices and edges. This redefinition doesn't change properties the graph determined by an adjacent and an incidence of its vertices and edges.

Fuzzy independent set, domination fuzzy set, fuzzy chromatic set are invariants concerning the isomorphism transformations of the fuzzy graphs and fuzzy hypergraph and allow make theirs structural analysis.

## BACKGROUND

The idea of fuzzy graphs has been introduced by Rosenfeld in a paper in (Zadeh, 1975), which has also been discussed in (Kaufmann, 1977).

The questions of using fuzzy graphs for cluster analysis were considered in (Matula,1970, Matula,1972). The questions of using fuzzy graphs in Database Theory were discussed in (Kiss,1991). The tasks of allocations centers on fuzzy graphs were considered in (Moreno, Moreno & Verdegay, 2001, Kutangila-Mayoya & Verdegay, 2005, Rozenberg & Starostina, 2005). The analyses and research of flows and vitality in transportation nets were considered in (Bozhenyuk, Rozenberg & Starostina, 2006). The fuzzy hypergraph applications to portfolio management, managerial decision making, neural cell-assemblies were considered in (Monderson & Nair, 2000). The using of fuzzy hypergraphs for decision making in CAD-Systems were also considered in (Malyshev, Bershtein & Bozhenyuk, 1991).

## MAIN DEFINITIONS OF FUZZY GRAPHS AND HYPERGRAPHS

This article presents the main notations of fuzzy graphs and fuzzy hypergraphs, invariants of fuzzy graphs and hypergraphs.
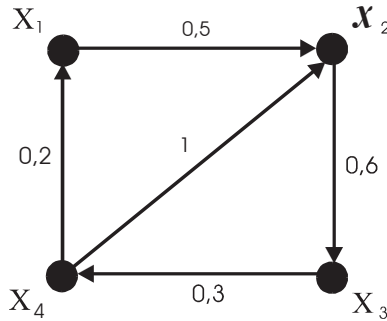
### Fuzzy Graph

Let a fuzzy direct graph $\widetilde{G} = (X, \widetilde{U})$ is given, where X is a set of **vertices**, $\widetilde{U} = \{\mu_U(x_i, x_j)|(x_i, x_j) \in X^2\}$ is a fuzzy set of **edges** with the membership function $\mu_U : X^2 \rightarrow [0,1]$ (Kaufmann, 1977).

Example 1. Let fuzzy graph $\widetilde{G}$ has X={$x_1, x_2, x_3, x_4$}, and $\widetilde{U}$ ={<0.5/($x_1,x_2$)>, <0.6/($x_2,x_3$)> , <0.3/($x_3,x_4$)> , <0.2/($x_4,x_1$)> , <1/($x_4,x_2$)>}. It is presented in figure 1.

The fuzzy graph $\widetilde{G}$ may present a fuzzy dependence relation between objects $x_1$, $x_2$, $x_3$, end $x_4$. If the object $x_j$ fuzzy depends from the object $x_i$, then there is direct **edge** ($x_i,x_j$) with membership function $\mu_U(x_i, x_j)$.

If a fuzzy relation, presented by fuzzy graph $\widetilde{G}$, is symmetrical, we have the fuzzy nondirect graph.

*Figure 1.*



A fuzzy graph $\widetilde{G} = (X, \widetilde{U})$ is convenient for representing as fuzzy **adjacent matrix** $\|r_{ij}\|_{n \times n}$, where $r_{ij} = \mu_U(x_i, x_j)$. So, the fuzzy graph, presented in figure 1, may be consider by adjacent matrix:

$$R_X = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ 0 & 0,5 & 0 & 0 \\ 0 & 0 & 0,6 & 0 \\ 0 & 0 & 0 & 0.3 \\ 0,2 & 1 & 0 & 0 \end{array}.$$

The fuzzy graph $\widetilde{H} = (X', \widetilde{U}')$ is called a fuzzy **subgraph** (Monderson & Nair, 2000) of $\widetilde{G} = (X, \widetilde{U})$ if $X' \subseteq X$ and $\widetilde{U}' \subseteq \widetilde{U}$.

Fuzzy directed **path** (Bershtein & Bozhenyuk, 2005) $\widetilde{L}(x_i, x_m)$ of graph $\widetilde{G} = (X, \widetilde{U})$ is called the sequence of fuzzy directed edges from vertex $x_i$ to vertex $x_m$:

$$\widetilde{L}(x_i, x_m) = <\mu_U(x_i, x_j)/(x_i, x_j)>, <\mu_U(x_j, x_k)/(x_j, x_k)>, ...$$
$$..., <\mu_U(x_1, x_m)/(x_1, x_m)>.$$

Conjunctive strength of path $\mu(\widetilde{L}(x_i, x_m))$ is defined as:

$$\mu(\widetilde{L}(x_i, x_m)) = \underset{<x_\alpha, x_\beta> ? \widetilde{L}(x_i, x_m)}{\&} \mu_U <x_\alpha, x_\beta>.$$

Fuzzy directed path $\widetilde{L}(x_i, x_m)$ is called simple path between vertices $x_i$ and $x_m$ if its part is not a path between the same vertices.

If a number of vertices $n \geq 3$ and $x_i = x_m$, then the path is called a cycle.

Obviously, what is it definition coincides with the same definition for nonfuzzy graphs.

Vertex $y$ is called fuzzy accessible from vertex $x$ in the graph $\widetilde{G} = (X, \widetilde{U})$ if exists a fuzzy directed path from vertex $x$ to vertex $y$.

The accessible degree of vertex $y$ from vertex $x$, $(x \neq y)$ is defined by expression:

$$\gamma(x,y) = \max_\alpha (\mu(\widetilde{L}_\alpha(x,y)), \quad \alpha = 1,2,....,p,$$

where $p$ - number of various simple directed paths from vertex $x$ to vertex $y$.

A subset of vertices $X'$ is called a fuzzy independent vertex set (Bershtein & Bozhenuk, 2001) with the degree of independence

$$\alpha(X') = 1 - \max_{? \, x_i, x_j ? \, X} \{\mu_U(x_i, x_j)\}.$$

A subset of vertices $X' \subseteq X$ of graph $\widetilde{G}$ is called a maximal fuzzy independent **vertex** set with the degree $\alpha(X')$, if the condition $\alpha(X'') < \alpha(X')$ is true for any $X' \subset X''$.

Let a set $\tau_k = \{X_{k1}, X_{k2}, ..., X_{kl}\}$ be given where $X_{ki}$ is a fuzzy independent $k$-vertex set with the degree of independent $\alpha_{ki}$. We define as

$$\alpha_k^{max} = \boldsymbol{max}\{\alpha_{x_k^1}, \alpha_{x_k^2}, ...., \alpha_{x_k^l}\}.$$

The value $\alpha_k^{max}$ means that fuzzy graph $\widetilde{G}$ includes k-vertex subgraph with the degree of independent $\alpha_k^{max}$ and doesn't include k-vertex subgraph with the degree of independence more than $\alpha_k^{max}$.

A fuzzy set

$$\Psi_X = \{<\alpha_1^{max}/1>, <\alpha_2^{max}/2>, ...., <\alpha_n^{max}/n>\}$$

is called a fuzzy **independent set** of fuzzy graph $\widetilde{G}$.

Fuzzy graph $\widetilde{G}$, presented in figure 1, has seven maximum fuzzy independent vertex sets:

$$\Psi_1 = \{x_2\}, \ \Psi_2 = \{x_4\}, \ \Psi_3 = \{x_1, x_3\}$$

with the degree of independence 1; $\Psi_4 = \{x_1, x_4\}$ with the degree of independence 0,8; $\Psi_5 = \{x_1, x_3, x_4\}$ with

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